arXiv:cs/0606001v1 [cs.DS] 1 Jun 2006

Tight Bounds on the Min-Max Boundary Decomposition Cost of Weighted Graphs

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February 1, 2008

Abstract

Many load balancing problems that arise in scientific computing applications boil down to the problem of partitioning a graph with weights on the vertices and costs on the edges into a given number of equally-weighted parts such that the *maximum boundary cost* over all parts is small.

Here, this partitioning problem is considered for graphs $G = (V, E)$ with edge costs $c\colon E \to \mathbb{R}_+$, that have bounded maximum degree and a *p-separator theorem* for some $p > 1$, i.e., any (arbitrarily weighted) subgraph of G can be separated into two parts of roughly the same weight by removing a *separator* $S \subseteq V$ such that the edges incident to S in the subgraph have total cost at most proportional to $(\sum_e c_e^p)^{1/p}$, where the sum is over all edges in the subgraph.

For arbitrary weights $w\colon V \to \mathbb{R}_+$, we show that the vertices of such graphs can be partitioned into k parts such that the weight of each part differs from the average weight $\sum_{v\in V} w_v/k$ by at most $(1-\frac{1}{k})\max_{v\in V} w_v$, and the boundary edges of each part have total cost at most proportional to $(\sum_{e\in E} c_e^p/k)^{1/p} + \max_{e\in E} c_e$. The partition can be computed in time nearly proportional to the time for computing separators S for G as above.

Our upper bound is shown to be tight up to a constant factor for infinitely many instances with a broad range of parameters. Previous results achieved this bound only if one has $c \equiv 1$, $w \equiv 1$, and one allows parts of weight as large as a constant multiple of the average weight.

We also give a separator theorem for d-dimensional grid graphs with arbitrary edge costs, which is the first result of its kind for non-planar graphs.

1 Introduction

We consider the problem to partition a weighted graph into a given number of parts subject to the constraint that the weight of each part differs from the average part weight only by a relatively small quantity. The objective is to minimize the *maximum boundary cost* over all parts, where the boundary cost of a part is the the total cost of the edges with exactly one endpoint in the part.

This problem naturally arises as a load balancing problem in scientific computing applications, where one wants to solve a large-scale problem given by a set V of jobs on a parallel computing system with k identical machines. The weight w_u is proportional to the time a machine takes to process job $u \in V$. However, job u may depend on other jobs $v \in V$. For each such dependency the graph $G = (V, E)$ contains an edge $e = \{u, v\} \in E$. If job v is not scheduled on the same machine as job u then a cost c_e is induced on the machines that handle jobs u and v. The cost c_e reflects the overhead for the communication needed to resolve the dependency among jobs u and v. How the makespan of a schedule increases under large communication costs, depends on the specific design of the considered parallel computing system. In general, one requires from a good schedule that the weights of the jobs are as equally distributed among the machines as possible and that the maximum communication cost over all machines is small. So this load balancing problem corresponds to the graph partitioning problem from above.

For example, consider the problem of large-scale climate simulation, where the surface of the earth is subdivided into many triangular regions. For each region, there is a job in V to simulate the weather in this region for a period of time. Of course, the simulations for neighboring regions depend on each other. So if jobs of neighboring regions are scheduled on distinct machines, one might have to interchange considerable amounts of data between the machines causing an increase of the makespan. This example also illustrates the use of weights and costs. Even if all regions have about the same area, the time for simulating the weather in these regions might differ tremendously depending on day-time, desired accuracy, et cetera. The degree of dependency among neighboring regions might differ in a similar manner.

Our aim is to characterize graph classes that, even for worst possible weights, allow k-way partitions that are good in the sense above, i.e., have equally-weighted parts and small boundary costs. We shall see that a “well-behaved” graph class allows good k-way partitions if and only if it allows good 2-way partitions, i.e., it has a *separator theorem*. In this, we can predict the scalability of the mentioned scientific computing applications.

Our results imply that there is no inherent trade-off between the weight-balancedness of a partition and its boundary costs. In particular, any partition, with weight of each part at most proportional to the average, can be transformed into a partition with almost equally-weighted parts such that the maximum boundary cost increases by at most constant factor, essentially.

Notice that the “quality” of a partition could also be measured by the average boundary cost instead of the maximum boundary cost. One might ask whether there are considerably better upper bounds for this measure than for the maximum boundary cost. We answer this question in the negative.

Previous Work and Contributions

Much work has been done on worst-case guarantees for graph partitioning problems. In a seminal article, Lipton and Tarjan [5] established a separator theorem for planar graphs, asserting that every n-vertex planar graphs can be separated into two parts of size at most $2n/3$ by removing $O(n^{1/2})$ vertices. Further separator theorems exist for graphs with an excluded minor [1] and for d-dimensional well-shaped meshes [7, 9, 6], where for the latter $O(n^{1-1/d})$ vertices can be removed instead of $O(n^{1/2})$. More generally, a graph is said to have a *p-separator theorem* (with respect to unit costs) if any induced subgraph can be separated into two parts of about the same weight by removing $O(n_0^{1/p})$ vertices, where n_0 is the number of vertices in the subgraph.

Simon and Teng [8] addressed the problem to partition a graph into $k \geq 2$ parts of weight at most proportional to the average, by removing edges from the graph. They showed that for bounded-degree graphs with a p-separator theorem such a partition can be achieved by removing $O(k^{1-1/p}n^{1/p})$ edges. So for unit edge-costs,the average boundary of the partition is at most proportional to $(n/k)^{1/p}$.

Kiwi, Spielman and Teng [4] were the first to give bounds on the maximum boundary cost instead of the average boundary cost. For unit-weights and unit-costs, they show that bounded-degree graphs with n vertices and p-separator theorem can be decomposed into k parts such that the weight of each parts is $O(n/k)$ and the maximum boundary cost is at most proportional to $(n/k)^{1/p}$. They also give bounds for partitions with maximum weight at most $(1+\epsilon)\cdot n/k$ and for the case of arbitrary weights. However, in these cases their bound on the maximum boundary cost increases by a factor $(1/\epsilon)^{1-1/p}$ and $(\log(k/\epsilon^2)/\epsilon)^{2-2/p}$, respectively. We show that this asymptotic increase of the maximum boundary cost can be avoided. More specifically, our bounds for the weighted case are the same as for the unweighted case, and in our results there is no trade-off between balancedness and boundary costs.

In Appendix A.3 we show that the obtained worst-case bounds on the maximum boundary cost are optimal with respect to the chosen parameters.

Strict weight-balancedness. It seems new to allow the constraint that the weight of each part may differ from the average weight of a part by at most $\frac{k-1}{k}\max_{v\in V} w_v$. Notice that this guarantee on the weight of the parts is the same as of an algorithm that assigns each vertex greedily to a part, i.e., a greedy bin-packing algorithm. However, in contrast to our methods, such a greedy algorithm will in general create huge boundary costs. In Section 5 we present a novel "shrink-and-conquer" algorithm that transforms any partition with loosely balanced weights into a strictly weight-balanced partition while maintaining the bounds on the maximum boundary cost. For the conquer-phase, a greedy bin-packing procedure is used. But our "shrink-and-conquer" approach shall ensure that this packing procedure touches every part only constantly often and therefore the boundary costs do increase only slightly in a conquer-phase.

Arbitrary edge costs. If one allows arbitrary costs $c\colon E\to\mathbb{R}_+$ on the edges instead of unit costs, then only the separator theorem for planar graphs [2] was known to extend to this case. Any bounded-degree planar graph can be separated into two parts of about the same weight by removing edges of cost $O((\sum c_e^2)^{1/2})$. In Section 6 we give a separator theorem for d-dimensional grid graphs. Every d-dimensional grid graph can be separated into two almost equally-weighted parts by removing edges of cost at most proportional to $(\sum c_e^{d/(d-1)})^{1-1/d}\cdot\log^{1/d}\phi$, where $\phi := \max_e c_e/\min_e c_e$ is the fluctuation of the edge costs. We think that the logarithmic factor in our grid separator theorem is superfluous. Moreover, we conjecture that many graph classes with separator theorem for unit costs also have a separator theorem for arbitrary c.

Assuming such separator theorems, we can extend the bounds on the maximum boundary cost of k-way partitions to the case of arbitrary edge costs. For this generalization, we utilize multi-balanced partitions (cf. Section 3), i.e., partitions that are simultaneously balanced with respect to several weight functions. Multi-balanced partitions were implicitly considered by Kiwi, Spielman, and Teng [4]. Their idea is to use recursive bisection where each separator divides the vertices evenly with respect to all weight functions. Such separators are increasingly difficult to find when the number of weight functions grows larger. Their approach gives the same guarantee as ours only if there are at most two weight functions. We use, instead of ordinary recursive bisection, a generalization thereof, which allows to balance the partition with respect to the weight functions one by one. This approach makes it possible to handle an arbitrary number of weight functions.

Notation

In this work, all considered graphs are assumed to be finite, undirected, and without self-loops or parallel edges. A graph $G = (V, E)$ has *size* $|G| := |V| + |E|$. For a subset $U \subseteq V$ of the vertices, $\delta(U) := \{e \in E \mid |e \cap U| = 1\}$ denotes the *cut* induced by U, or the set of *boundary edges* of U. We let $G[W] := (W, E(W))$ be the graph induced by a vertex set $W \subseteq V$ in G, where $E(W) := \{e \in E \mid e \subseteq W\}$ is the set of edges running in W. For all other graph notations, we refer to any standard text book on graph theory or algorithms.

Let $f\colon X \to \mathbb{R}_+$ be a function on a finite domain X. If not ambiguous, we write $f_x := f(x)$. For a subset $S \subseteq X$, we define $f(S) := \sum_{s\in S} f_s$. For $p > 1$, the *p-norm* of f is given by $\|f\|_p := (\sum_{x\in X} f_x^p)^{1/p}$. In the limit, we have $\|f\|_\infty = \max_{x\in X} f_x$. *Hölder's inequality* states that $\sum_{x\in X} f_x g_x \leq \|f\|_p \cdot \|g\|_q$ for functions $f, g\colon X \to \mathbb{R}_+$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We denote the *restriction* of f to $S \subseteq X$ by $f_{|S}\colon S \to \mathbb{R}_+$ with $f_{|S}(x) := f(x)$ for all $x \in S$. The function on domain X that is identical to 1 is denoted by $\mathbb{1}_X : X \to \{1\}$. When adding two non-negative function f and g with respective domains X and Y, we implicitly extend f and g to the domain $X \cup Y$ with $f_{|Y\setminus X} \equiv 0$ and $g_{|X\setminus Y} \equiv 0$. Then we can define $f + g : X \cup Y \to \mathbb{R}_+$ by $(f + g)(x) = f(x) + g(x)$. When the domains X and Y are disjoint we call the sum $f + g$ *direct* and write sometimes $f \oplus g$.

We write $f = O_d(g)$, $f \ll_d g$, or $g = \Omega_d(f)$ for expressions f and g if $|f| \leq C \cdot |g|$ for some constant C that might depend on a parameter d.

2 Min-Max Boundary Decomposition Problem

In the following we provide a formalization of this problem and state our main results. The decomposition problem is formulated in terms of vertex colorings instead of partitions, since we find this notation more convenient.

Definition 1 (Strictly Balanced Colorings). Let k be a positive integer and $G = (V, E)$ be a graph with edge costs $c\colon E \to \mathbb{R}_+$ and vertex weights $w\colon V \to \mathbb{R}_+$.

A *k-coloring* $\chi\colon V \to [k]$ of G is *strictly (w-)balanced* if the weight of each *color class* $\chi^{-1}(i) := \{v \in V \mid \chi(v) = i\}$ differs from the average weight of a color class by no more than a $(1 - \frac{1}{k})$-fraction of $\|w\|_\infty$, i.e., when we have

$$\max_{1\leq i\leq k} \left| w(\chi^{-1}(i)) - \frac{\|w\|_1}{k} \right| \leq (1 - 1/k) \cdot \|w\|_\infty. \tag{1}$$

The *maximum boundary cost* of a coloring χ of G is defined as the maximum cost of the boundary edges $\delta(\chi^{-1}(i))$ of a color class $\chi^{-1}(i)$, formally,

$$\|\partial\chi^{-1}\|_\infty := \max_{1\leq i\leq k} c(\delta(\chi^{-1}(i))).$$

(The strange symbol $\|\partial\chi^{-1}\|_\infty$ will be consistent with our further notation.)

Notice that strictly balanced coloring are as weight-balanced as possible for many parameter choices. More precisely, for all choices of k and $\|w\|_\infty$, there are infinitely many choices of $\|w\|_1$ such that equality holds in (1) even for the most weight-balanced coloring of some instance with the chosen parameters.

For the applications mentioned in Section 1, it is desirable to know which graphs allow strictly balanced k-colorings of small maximum boundary cost even if the weights of the vertices are chosen adversarial.

Definition 2. The *min-max boundary (k-)decomposition cost* of G with edge costs $c\colon E \to \mathbb{R}_+$ is the minimum maximum boundary cost over all strictly balanced k-colorings of G with respect to worst possible weights, formally,

$$\partial_\infty^k(G,c) := \sup_{w\colon V\to\mathbb{R}_+} \min_{\chi} \|\partial\chi^{-1}\|_\infty,$$

where the minimum is over all strictly w-balanced k-colorings χ of G.

In our main theorem we will upper bound $\partial_\infty^k(G,c)$ in terms of a parameter related to ∂_∞^2 that is subgraph-monotone, i.e., it does not increase when going to (induced) subgraphs. Note that the trivial subgraph-monotone version of ∂_∞^2, namely

$$\max_{W\subseteq V} \partial_\infty^2(G[W], c_{|E(W)}),$$

is pointless, since it gives no information about how large the costs for decomposing $G[U]$ are compared to the edge costs $c_{|E(U)}$ in this subgraph. For a meaningful parameter, we need to relate the decomposition cost of a subgraph to the costs of its edges.

The following definition formalizes this idea.

Definition 3 (Splitting Sets, p-Splittablity). For any *splitting value* w^* with $0 \le w^* \le \|w\|_1$, a vertex set $U \subseteq V$ is said to be w^**-splitting* if $|w(U) - w^*| \le \|w\|_\infty/2$. The *(boundary) cost* of U is $\partial U := c(\delta(U))$.

Now the *p-splittability* of a graph G with edge costs c is the least number $\sigma_p(G,c)$ such that for every induced subgraph $G[W]$, all weights $w\colon W \to \mathbb{R}_+$ and splitting values w^*, there exists a w^*-splitting set $U \subseteq W$ with boundary cost $\partial_W U \le \sigma_p(G,c)\cdot\|c_{|W}\|_p$, where $\partial_W U$ is the boundary cost of U in $G[W]$, and $c_{|W}$ denotes the restriction of c to the edges of G running in W. We write $\sigma_p := \sigma_p(G,c)$ when G and c are understood.

We remark that if instance (G,c) is *well-behaved*, i.e., G has bounded maximum degree and $c(u,v) = \Omega(c(u,v'))$ for all edges $\{u,v\},\{u,v'\} \in E$, then it holds

$$\sigma_p(G,c) = \Theta(\max_{W\subseteq V} \partial_\infty^2(G[W], c_{|E(W)})/\|c_{|E(W)}\|_p)$$

(cf. Corollary 39). So we can indeed view parameter σ_p as a subgraph-monotone version of ∂_∞^2. However, it is much more convenient to work with splitting sets instead of strictly balanced 2-colorings.

Our main theorem gives an upper bound on the min-max boundary decomposition cost in terms of the p-splittability and the *maximum c-weighted degree* $\Delta_c := \max_{v\in V} c(\delta(v))$. The time for computing k-colorings that achieve the bounds of the theorem is almost the same as for computing cheap splitting sets in the graph.

Theorem 4. *Let G be a graph with edge costs c. Then for all $k \in \mathbb{N}$ and $p > 1$,*

$$\partial^k_\infty(G, c) = O_p(\sigma_p \cdot (k^{-1/p} \cdot \|c\|_p + \Delta_c)).$$

Moreover, suppose one can compute splitting sets of cost at most $s \cdot \|c_{|W}\|_p$ in time $t(|G[W]|)$ for all subgraphs $G[W]$, weights w and splitting values w^*, where $t(n) \geq n$ is a linear function in n. Then, there exists an $O(t(|G|) \cdot \log k)$-time algorithm to compute strictly balanced k-colorings of G with maximum boundary cost $O_p(s \cdot (k^{-1/p} \cdot \|c\|_p + \Delta_c))$.

In the remainder of this section we draw a connection between the above result and the more common notion of "separator theorems", which were already mentioned in Section 1. The connection also allows us to formulate a result asserting the tightness of our upper bound on the min-max boundary decomposition cost.

In the following, we will assume that the considered instances (G, c) consisting of a graph $G = (V, E)$ and edge costs $c\colon E \to \mathbb{R}_+$ are *well-behaved*, i.e., the maximum degree $\Delta(G)$ is bounded and the local fluctuation $c(\delta(v))/\min_{e\in\delta(v)} c(e)$ at each vertex $v \in V$ is bounded. In Appendix A.3 we discuss to what extend this assumption is necessary.

Before formulating our results, we need to introduce a few notions (cf. Appendix A.3). A subset $S \subseteq V$ of the vertices is a *balanced separator* with respect to weights $w\colon V \to \mathbb{R}_+$ if all components of $G[V \setminus S]$ have weight at most $\frac{2}{3}\|w\|_1$. The *cost* of S is the total cost $\sum_{s\in S} c(\delta(s))$ of the edges incident to S. Then, a well-behaved instance (G, c) has a p-separator theorem (cf. Definition 35) if for all $W \subseteq V(G)$ and weights $w\colon W \to \mathbb{R}_+$ there exists w-balanced separators in $G[W]$ of cost $O(\|c_{|W}\|_p)$. Since well-behaved instances with p-separator theorem have constant p-splittability (cf. Lemma 37), our main theorem translates to graphs with separator theorems.

Roughly speaking, the theorem below shows that a well-behaved graph class, which is closed under a reasonable "similarity" relation, has small min-max boundary k-decomposition cost if and only if it has a p-separator theorem for some $p > 1$.

Theorem 5. *Let (G, c) be a well-behaved instance. If (G, c) has a p-separator theorem, then we have for all $k \in \mathbb{N}$,*

$$\partial^k_\infty(G, c) = O_p(\|c\|_p/k^{1/p} + \|c\|_\infty).$$

If there exists a weight function $w\colon V \to \mathbb{R}_+$ with $\|w\|_\infty \leq \|w\|_1/4$ such that all w-balanced separators of G have cost $\alpha \cdot \|c\|_p$, then for infinitely many $k \in \mathbb{N}$ there exist instances $(\tilde{G}, \tilde{c})$ "similar" to (G, c) with

$$\partial^k_\infty(\tilde{G}, \tilde{c}) \gg \alpha \cdot (\|\tilde{c}\|_p/k^{1/p} + \|\tilde{c}\|_\infty).$$

In the theorem above, instance $(\tilde{G}, \tilde{c})$ is similar to (G, c) in the sense that $\tilde{G}$ is the union of $k/4$ disjoint isomorphic copies of G, and $\tilde{c}$ assumes for an edge in $\tilde{G}$ the cost of the corresponding edge in G. We remark that there are weights $\tilde{w}$ for $\tilde{G}$ such that every k-coloring χ of $\tilde{G}$ with roughly balanced weights, i.e., $\max_i \tilde{w}(\chi^{-1}(i)) \leq 2\|\tilde{w}\|_1/k$, has average boundary cost at least proportional to $\alpha \cdot (\|\tilde{c}\|_p/k^{1/p} + \|\tilde{c}\|_\infty)$. So we cannot expect better general upper bounds than in Theorem 5, even if we relax the strict balancedness constraint and consider the average instead of the maximum boundary cost.

For the proof of Theorem 5 we refer to Appendix A.3. Notice that the first part is implied by Theorem 4 and the fact that σ_p is at most a constant for well-behaved graphs with p-separator theorem. The second part follows from the observation that a balanced separator of G can be constructed from each restriction $\chi_{|G^{(i)}}$ of a roughly balanced coloring χ of $\tilde{G}$ to one of the copies of G, say $G^{(i)}$.

In Sections 3-5 we sketch a proof of Theorem 4. First, it is instructive to introduce further notation that allows to formulate our results and proofs more easily.

Further Notation. Let $\Phi\colon V \to \mathbb{R}_+$ be a non-negative function on the vertices of a graph $G = (V, E)$. We extend Φ on the power set of V implicitly by the notation $\Phi(U) := \sum_{u \in U} \Phi(u)$ for $U \in 2^V$. So it is justified to call Φ a *measure* on V.

Let χ be a k-coloring of G. The function $\Phi\chi^{-1}\colon [k] \to \mathbb{R}_+$ with $(\Phi\chi^{-1})(i) := \Phi(\chi^{-1}(i))$ maps each color to the *Φ-measure* or *Φ-weight* of its color class. So, $\|\Phi\chi^{-1}\|_\infty$ is the *maximum Φ-measure* (of a color class) of χ.

For any non-negative discrete function f, we write $\|f\|_{avg} := \|f\|_1/k$ when the number k of colors is understood. So, $\|\Phi\chi^{-1}\|_{avg}$ is the *average Φ-measure* (of the color classes) of χ and we have $\|\Phi\chi^{-1}\|_{avg} = \|\Phi\|_1/k = \|\Phi\|_{avg}$.

Analogously, the function $\partial\chi^{-1}\colon [k] \to \mathbb{R}_+$ with $(\partial\chi^{-1})(i) := \partial(\chi^{-1}(i))$ maps each color to the boundary cost of its color class. So, $\|\partial\chi^{-1}\|_\infty$ is the *maximum boundary cost* of χ as in Definition 1 and $\|\partial\chi^{-1}\|_{avg}$ is the *average boundary cost* of coloring χ. Clearly, $\|\partial\chi^{-1}\|_{avg} \leq \|\partial\chi^{-1}\|_\infty$.

For disjoint vertex sets $W_0, W_1 \subseteq V$, we can combine two k-colorings $\chi_0\colon W_0 \to [k]$ and $\chi_1\colon W_1 \to [k]$ into a k-coloring of $W := W_0 \cup W_1$, by forming the *direct sum* $\chi_0 \oplus \chi_1\colon W \to [k]$ with $(\chi_0 \oplus \chi_1)(i) = \chi_b(i)$ if $i \in W_b$.

3 Multi-balanced colorings

In this section, we relax the strict constraints on the weights of the color classes and consider (non-strictly) balanced colorings.

We say that a coloring χ of $G = (V, E)$ is *(weakly) balanced* with respect to a vertex measure $\Phi\colon V \to \mathbb{R}_+$ if $\|\Phi\chi^{-1}\|_\infty = O(\|\Phi\|_{avg} + \|\Phi\|_\infty)$.

Furthermore, we are interested in colorings that are not only balanced with respect to a single measure Φ like the weights w, but with respect to a constant number of measures $\Phi^{(1)}, \dots, \Phi^{(r)}$. We call such colorings *multi-balanced*. We shall see that the proof of Theorem 4 greatly benefits from the following results about multi-balanced colorings.

The main result of this section is the following lemma, which provides a bound on the minimum average boundary cost of multi-balanced k-colorings.

Lemma 6 (Multi-bal. Min-Avg. Boundary). *Let $G = (V, E)$ be a graph with edge costs $c\colon E \to \mathbb{R}_+$ and vertex measures $\Phi^{(1)}, \dots, \Phi^{(r)}$. Then, there exists a k-coloring χ that is balanced with respect to $\Phi^{(1)}$ through $\Phi^{(r)}$ and has average boundary cost at most proportional to $\sigma_p \cdot k^{-1/p} \cdot \|c\|_p$. More precisely, we have*

$$\|\Phi^{(j)}\chi^{-1}\|_\infty = O_r(\|\Phi^{(j)}\|_{avg} + \|\Phi^{(j)}\|_\infty) \text{ for } j \in [r]$$
$$\|\partial\chi^{-1}\|_{avg} = O_r(\sigma_p \cdot q \cdot k^{-1/p} \cdot \|c\|_p)$$

with $1 = \frac{1}{p} + \frac{1}{q}$. One can compute such a coloring in time $O_r(t(|G|) \cdot \log k)$ with t as in Theorem 4.

We remark that one can in fact guarantee $\|\Phi^{(1)}\chi^{-1}\|_\infty \le 3\|\Phi^{(1)}\|_{avg} + O_r(\|\Phi^{(1)}\|_\infty)$ for the coloring obtained by Lemma 6.

Similar to [4], we observe that in the proof of Lemma 6 the boundary cost function $\partial : 2^V \to \mathbb{R}_+$, which assigns each subset U of V its boundary cost ∂U, can approximately be modeled as a vertex measure. Hence, the boundary cost of a coloring can be balanced by the methods developed for Lemma 6. This idea yields the following proposition, which makes up the first out three steps towards Theorem 4.

Proposition 7 (Multi-bal. Min-Max Boundary). *Let G be as in Lemma 6. Then, there exists a k-coloring χ which is balanced with respect to $\Phi^{(1)}$ through $\Phi^{(r)}$ and has maximum boundary cost at most proportional to $\sigma_p \cdot (k^{-1/p} \cdot \|c\|_p + \Delta_c)$, formally,*

$$\|\Phi^{(j)}\chi^{-1}\|_\infty = O_r(\|\Phi^{(j)}\|_{avg} + \|\Phi^{(j)}\|_\infty) \text{ for } j \in [r]$$
$$\|\partial\chi^{-1}\|_\infty = O_r(\sigma_p \cdot (q \cdot k^{-1/p} \cdot \|c\|_p + \Delta_c)).$$

One can compute such a coloring in time $O_r(t(|G|) \cdot \log k)$ with t as in Theorem 4.

In order to show Lemma 6, we need the following auxiliary lemma, which itself can be viewed as a refined version of Lemma 6 for the case $k = 2$.

Lemma 8. *Let G be as in Lemma 6. Then, each vertex set $W \subseteq V$ can be 2-colored such that the cost of the edges between the two color classes is at most $(2^r - 1) \cdot \sigma_p(G, c) \cdot \|c_{|W}\|_p$ and for all $j \in [r]$, the $\Phi^{(j)}$-measure of each color class does not exceed $\frac{3}{4}(\Phi^{(j)}(W) + 2^{r-j}\|\Phi^{(j)}\|_\infty)$. In particular, the $\Phi^{(1)}$-measure of each color class is at most $\frac{1}{2}(\Phi^{(1)}(W) + 2^{r-1}\|\Phi^{(1)}\|_\infty)$.*

Furthermore, such a 2-coloring of $G[W]$ can be found in time $O_r(t(|G[W]|))$ where t is as in Theorem 4.

Proof. By induction on $r \ge 1$. First, graph G is bisected into two parts U_1 and U_2 with respect to measure $\Phi^{(r)}$. More precisely, from the definition of $\sigma_p(G, c)$ it follows that there exists a splitting set $U_1 \subseteq W$ with cost at most $\partial_W U_1 \le \sigma_p(G, c) \cdot \|c_{|W}\|_p$ such that

$$|\Phi^{(r)}(U_1) - \Phi^{(r)}(W)/2| \le \|\Phi^{(r)}\|_\infty/2 \tag{2}$$

Let $U_2 := W \backslash U_1$ be the complement of U_1 within W. In the case $r = 1$, the coloring $\chi \colon W \to \{1, 2\}$ with $\chi_{|U_b} \equiv b$ fulfills all requirements of the lemma.

Therefore, we may assume $r > 1$. By induction hypothesis, we can find 2-colorings χ_1 and χ_2 of $G[U_1]$ and $G[U_2]$, that fulfill the conditions of the lemma for $\Phi^{(1)}$ through $\Phi^{(r-1)}$, i.e., for $b \in \{1, 2\}$ and $j \in [r-1]$,

$$\|\Phi^{(j)}\chi_b^{-1}\|_\infty \le 3/4(\Phi^{(j)}(U_1) + 2^{r-1-j}\|\Phi^{(j)}\|_\infty) \tag{3}$$
$$\|\partial\chi_b^{-1}\|_\infty \le (2^{r-1} - 1) \cdot \sigma_p(G, c) \cdot \|c_{|U_b}\|_p \tag{4}$$

Without loss of generality, we may assume that for $b \in \{1, 2\}$, that

$$\Phi^{(r)}\chi_b^{-1}(b) \le \tfrac{1}{2}\Phi^{(r)}(U_b) \tag{5}$$

Now, let $\chi\colon W \to \mathbb{R}_+$ be the direct sum of χ_1 and χ_2, i.e., $\chi_{|U_b} = \chi_b$. Then

$$\begin{aligned}
\Phi^{(r)}\chi^{-1}(b) &= \Phi^{(r)}\chi_1^{-1}(b) + \Phi^{(r)}\chi_2^{-1}(b) \\
&\overset{(5)}{\leq} 1/2 \cdot \Phi^{(r)}(U_b) + \Phi^{(r)}(W \backslash U_b) \\
&\overset{(2)}{\leq} 1/4 \cdot \Phi^{(r)}(W) + 1/2 \cdot \Phi^{(r)}(W) + 3/4 \cdot \|\Phi^{(r)}\|_\infty \\
&= 3/4 \cdot (\Phi^{(r)}(W) + 2^{r-r}\|\Phi^{(r)}\|_\infty)
\end{aligned}$$

We maintain for $j < r$

$$\|\Phi^{(j)}\chi^{-1}\|_\infty \leq \sum_b \|\Phi^{(j)}\chi_b^{-1}\|_\infty \overset{(3)}{\leq} \tfrac{3}{4}(\Phi^{(j)}(W) + 2^{r-j}\|\Phi^{(j)}\|_\infty)$$

Similarly, we have $\|\partial\chi^{-1}\|_\infty \leq \partial_W U_1 + \sum_b \|\partial\chi_b^{-1}\|_\infty \overset{(4)}{\leq} (2^r - 1) \cdot \sigma_p(G, c) \cdot \|c_{|W}\|_p$ using the fact that $\|c_{|U_b}\|_p \leq \|c_{|W}\|_p$ for $b \in \{1, 2\}$. The additional guarantee

$$\|\Phi^{(1)}\chi^{-1}\|_\infty \leq \sum_b \|\Phi^{(1)}\chi_b^{-1}\|_\infty \leq \tfrac{1}{2}(\Phi^{(j)}(W) + 2^{r-1}\|\Phi^{(1)}\|_\infty)$$

is also easily seen to be maintained. □

The proof of Lemma 6 is by induction on the number of measures to be balanced. Due to the length of the proof, we formulate the induction step as a lemma of its own. It states that given any coloring χ, one can efficiently compute a new coloring $\hat{\chi}$ which is balanced with respect to the measure $\Phi^{(1)}$ such that the maximum $\Phi^{(j)}$-measure $(1 < j \leq k)$ and the average boundary cost of the coloring increases by essentially at most a constant factor. So if χ was balanced with respect to measure $\Phi^{(2)}$ to $\Phi^{(r)}$, then $\hat{\chi}$ is balanced with respect to measures $\Phi^{(1)}$ to $\Phi^{(r)}$.

Lemma 9. *Let G be as in Lemma 6 and let χ be an arbitrary k-coloring of G.*
Then, a k-coloring $\hat{\chi}$ of G can be found in time $O_r(t(|G|)\log k)$ such that

$$\begin{aligned}
\|\Phi^{(1)}\,\hat{\chi}^{-1}\|_\infty &= O_r(\|\Phi^{(1)}\|_{avg} + \|\Phi^{(1)}\|_\infty) \\
\|\Phi^{(j)}\hat{\chi}^{-1}\|_\infty &= O_r(\|\Phi^{(j)}\chi^{-1}\|_\infty + \|\Phi^{(j)}\|_\infty) \\
\|\partial\hat{\chi}^{-1}\|_{avg} &= O_r(\|\partial\chi^{-1}\|_\infty + \mathcal{B})
\end{aligned}$$

with $\mathcal{B} = q \cdot k^{-1/p} \cdot \sigma_p \cdot \|c\|_p$, and t as in Theorem 4.

Since the induction basis, $r = 0$, for Lemma 6 is trivial, it only remains to show Lemma 9 and Proposition 7.

Proof of Lemma 9. During the construction of $\hat{\chi}$, for each color $i \in [k]$ a *tentative* color class $tent(i) \subseteq V$ is maintained. We start with $tent(i) = \chi^{-1}(i)$ for each color i. The algorithm has the invariant:

Invariant 1. $\{tent(i)\}_{i\in[k]}$ is a partition of V.

In fact, $\mathit{tent}(i)$ will assume at most three different sets in the course of the algorithm for each color i. For convenience, let $\Psi := \Phi^{(1)}$. According to the Ψ-weight of $\mathit{tent}(i)$, we maintain a partition of the color set $[k]$.

$$\begin{aligned} \mathit{Light} &= \{i \in [k] \mid \Psi(\mathit{tent}(i)) < \|\Psi\|_{avg}\} \\ \mathit{Heavy} &= \{i \in [k] \mid \Psi(\mathit{tent}(i)) \geq 3\|\Psi\|_{avg} + 2^r\|\Psi\|_\infty\} \\ \mathit{Medium} &= [k] \setminus (\mathit{Light} \cup \mathit{Heavy}) \end{aligned}$$

Each heavy color $i \in [k]$ will, in some iteration of our algorithm, be turned into a medium color by MOVEing vertices from $\mathit{tent}(i)$ to tentative color classes of light colors.

Invariant 1 and the definition of the partition $\{\mathit{Light}, \mathit{Medium}, \mathit{Heavy}\}$ imply the claim below, asserting that we can assign at least two distinct light colors to every heavy color.

Claim 1. $|\mathit{Light}| \geq 2|\mathit{Heavy}|$.

Proof: We have $\|\Psi\|_{avg}|Medium| + 3\|\Psi\|_{avg}|Heavy| \leq \|\Psi\|_1 = k \cdot \|\Psi\|_{avg}$. Since $k = |\mathit{Light}| + |\mathit{Medium}| + |Heavy|$, we get $2|Heavy| \leq |\mathit{Light}|$. ∎

We have another partition of $[k]$ into parts *Untouched*, *Pending*, and *Finished*. Initially all colors are untouched. As the names suggest, we will have $\mathit{tent}(i) = \chi^{-1}(i)$ for untouched colors and $\mathit{tent}(i) = \hat{\chi}^{-1}(i)$ for finished colors. For pending colors, $\mathit{tent}(i)$ is a common superset of both $\chi^{-1}(i)$ and $\hat{\chi}^{-1}(i)$, and so we might be obliged to change tentative color classes of pending colors.

At each point in the algorithm, the two color partitions that we maintain are related to each other in the following way.

Invariant 2. The following inclusions hold:

$$\mathit{Light} \subseteq \mathit{Untouched} \qquad \mathit{Heavy} \subseteq \mathit{Pending}$$
$$\mathit{Finished} \subseteq \mathit{Medium}$$

To set up the inclusion $\mathit{Heavy} \subseteq \mathit{Pending}$ in the beginning, we let each "initially" heavy color i with $\Psi\chi^{-1}(i) \geq 3\|\Psi\|_{avg} + 2^r\|\Psi\|_\infty$ be *Pending*.

As indicated before, an iteration of our algorithm consists of MOVEing vertices from one tentative color class to other tentative color classes.

For each color $i \in [k]$, we will have a set $V_{in}(i)$ of vertices *in*coming to color i and a set $V_{out}(i)$ of vertices *out*going of color i. Similar to a network flow conservation law, we have

$$\chi^{-1}(i) \cup V_{in}(i) = \hat{\chi}^{-1}(i) \cup V_{out}(i). \tag{6}$$

None of $V_{in}(i)$, $\hat{\chi}^{-1}(i)$, or $V_{out}(i)$ is a "dynamic" sets. Once defined by the algorithm, the sets are not changed afterwards. On the other hand, the two partitions of $[k]$ and the tentative color classes may change during the construction. As the names suggest, it is not possible that a color gets *Untouched* again after it was pending or even finished.

We can now define the tentative color class $\mathit{tent}(i)$ in terms of $\chi^{-1}(i)$, $V_{in}(i)$, and $\hat{\chi}^{-1}(i)$ depending on the current *state* of color i,

$$\mathit{tent}(i) = \begin{cases} \chi^{-1}(i) & \text{if } i \in \mathit{Untouched}, \\ \chi^{-1}(i) \,\dot{\cup}\, V_{in}(i) & \text{if } i \in \mathit{Pending}, \\ \hat{\chi}^{-1}(i) & \text{if } i \in \mathit{Finished}. \end{cases}$$

Then, our algorithm consists of iterating the following procedure as long as there exist pending colors. (Remember that we start with $\mathit{Pending} = \{i \in [k] \mid \Psi\chi^{-1}(i) \geq 3\|\Psi\|_{avg} + 2^r\|\Psi\|_\infty\}$ and $\mathit{Finished} = \emptyset$.)

Procedure Move (color $i \in [k]$)

// *Precondition: color i is pending.*

(1.) If $i \in \mathit{Medium}$,

then augment $\hat{\chi}$ such that $\hat{\chi}^{-1}(i) = \mathit{tent}(i)$,

move i from *Pending* to *Finished*; return.

// *If pending color i is not medium, then $i \in \mathit{Heavy}$ by Invariant 2, and so $|\mathit{Light}| > 2$ by Claim 1.*

(2.) Choose distinct colors $x_1, x_2 \in \mathit{Light}$.

(3.) Compute a splitting set U in $G[X]$ of cost $\partial_X U \leq \sigma_p \cdot \|c_{|X}\|_p$, where $X := X(i) := \mathit{tent}(i)$ with $\|\Psi\|_{avg} \leq \Psi(U) \leq \|\Psi\|_{avg} + \|\Psi\|_\infty$.

(4.) Find a 2-coloring χ_0 of $G[W]$ as in Lemma 8 where $W := V_{out}(i) := X \backslash U$.

(5.) Augment coloring $\hat{\chi}$ such that $\hat{\chi}^{-1}(i) = U$ and define $V_{in}(x_b) := \chi_0^{-1}(b)$ for $b \in \{1, 2\}$.

(6.) Move color i from *Pending* to *Finished* and colors x_1, x_2 from *Untouched* to *Pending*; return.

We observe that for i, x_1, x_2 as above, it holds

$$V_{out}(i) = V_{in}(x_1) \cup V_{in}(x_2) \tag{7}$$

The procedure Move maintains Invariant 1. From the claim below it follows that also Invariant 2 is maintained. The claim holds since both color classes of χ_0 have Ψ-weight at least $\|\Psi\|_{avg}$ for i being heavy.

Claim 2. If procedure Move is applied to a heavy color i and colors x_1, x_2 are selected in step (2.), then one has $i \in \mathit{Medium}$ and $x_1, x_2 \notin \mathit{Light}$ afterwards.

Proof: After the procedure, color i is medium because $\Psi(\mathit{tent}(i)) = \Psi(U)$ and $\|\Psi\|_{avg} \leq \Psi(U) \leq \|\Psi\|_{avg} + \|\Psi\|_\infty$ by construction. From Lemma 8, we get $\Psi V_{in}(x_b) \geq 1/2 \cdot (\Psi(W) - 2^{r-1}\|\Psi\|_\infty)$. By construction, we have $\Psi(W) = \Psi(X) - \Psi(U)$ and $\Psi(U) \leq \|\Psi\|_{avg} + \|\Psi\|_\infty$. Since color i was heavy when procedure Move was applied, we have $\Psi(X) \geq 3\|\Psi\|_{avg} + 2^r\|\Psi\|_\infty$ and therefore

$$\begin{aligned}\Psi V_{in}(x_b) &\geq 1/2 \cdot (\Psi(X) - \Psi(U) - 2^{r-1}\|\Psi\|_\infty) \\ &\geq 1/2 \cdot (\Psi(X) - \|\Psi\|_{avg} - \|\Psi\|_\infty - 2^{r-1}\|\Psi\|_\infty) \\ &\geq 1/2 \cdot (2\|\Psi\|_{avg}) = \|\Psi\|_{avg}.\end{aligned}$$

By construction, each color x_b is *Pending* after Move(i) and hence $\Psi(\mathit{tent}(x_b)) \geq \Psi V_{in}(x_b) \geq \|\Psi\|_{avg}$. Thus, color x_b cannot be *Light* anymore. ■

After termination of the algorithm, we have $\mathit{Pending} = \emptyset$ and by Invariant 2 also $\mathit{Heavy} = \emptyset$. So we obtain a Ψ-balanced coloring when we let the color classes of $\hat{\chi}$ agree with the tentative color classes. We show next that our construction increased the maximum $\Phi^{(j)}$-measure by not more than a constant factor (essentially).

The procedure Move induces a tree-like structure on the set of colors. More specifically, let $\mathcal{F}$ be the directed binary forest on nodes $[k]$ where a node $i \in [k]$ has children $x_1, x_2 \in [k]$ if $V_{out}(i) = V_{in}(x_1) \cup V_{in}(x_2)$.

By Lemma 8 and identity (7) we have for each arc (i,x) in $\mathcal{F}$, that $\Phi^{(j)}V_{in}(x) \leq 3/4 \cdot \Phi^{(j)}V_{out}(i) + O_r(\|\Phi^{(j)}\|_\infty)$. This observation and identity (6) imply that the $\Phi^{(j)}$-weight of $V_{in}(i)$ decreases geometrically along the arcs of $\mathcal{F}$, i.e.,

$$\Phi^{(j)}V_{in}(x) \leq \tfrac{3}{4}\Phi^{(j)}V_{in}(i) + O_r(\|\Phi^{(j)}\chi^{-1}\|_\infty + \|\Phi^{(j)}\|_\infty) \tag{8}$$

Since $V_{in}(s) = \emptyset$ for each root s of $\mathcal{F}$ and since the geometric series over 3/4 is convergent, relation (8) implies $\Phi^{(j)}V_{in}(i) = O_r(\|\Phi^{(j)}\chi^{-1}\|_\infty + \|\Phi^{(j)}\|_\infty)$. So the claim below holds for each color $i \in [k]$, because $\hat{\chi}^{-1}(i) \subseteq X(i) := \chi^{-1}(i) \cup V_{in}(i)$ by identity (6),

Claim 3. $\Phi^{(j)}\hat{\chi}^{-1}(i) \leq \max_i \Phi^{(j)}X(i) = 4\|\Phi^{(j)}\chi^{-1}\|_\infty + O_r(\|\Phi^{(j)}\|_\infty)$.

Proof: By induction on the distance h of color i from a root s in $\mathcal{F}$, we conclude from relation (8) that

$$\Phi^{(j)}V_{in}(i) \leq (3/4)^h\Phi^{(j)}V_{in}(s) + \sum_{l=1}^{\infty}(3/4)^l(\|\Phi^{(j)}\chi^{-1}\|_\infty + O_r(\|\Phi^{(j)}\|_\infty)).$$

Since $V_{in}(s) = \emptyset$, we have

$$\Phi^{(j)}V_{in}(i) \leq 3\|\Phi^{(j)}\chi^{-1}\|_\infty + O_r(\|\Phi^{(j)}\|_\infty).$$

By identity (6), we have $\Phi^{(j)}\hat{\chi}^{-1}(i) \leq \Phi^{(j)}X(i) = \Phi^{(j)}V_{in}(i) + \|\Phi^{(j)}\chi^{-1}\|_\infty$ and therefore $\|\Phi^{(j)}\hat{\chi}^{-1}\|_\infty \leq \max_i \Phi^{(j)}X(i) \leq 4\|\Phi^{(j)}\chi^{-1}\|_\infty + O_r(\|\Phi^{(j)}\|_\infty)$. ■

What remains is to estimate the average boundary cost of the coloring $\hat{\chi}$ in terms of $\|\partial\chi^{-1}\|_{avg}$ and $\mathcal{B}$. By Lemma 8, the cost of the edges cut by Move (i) is at most proportional to $\sigma_p \cdot \|c_{|X(i)}\|_p$. So we have

$$\|\partial\hat{\chi}^{-1}\|_{avg} \leq \|\partial\chi^{-1}\|_{avg} + O_r(\sigma_p \textstyle\sum_{i=1}^{k} \|c_{|X(i)}\|_p/k) \tag{9}$$

To meet the requirements of the lemma, we need to show that $\sum_{i=1}^{k} \|c_{|X(i)}\|_p = O_r(\mathcal{B})$. The idea is to consider first each component of $\mathcal{F}$ separately. Let $C_s \subseteq [k]$ denote the tree component of $\mathcal{F}$ with root $s \in [k]$. We shall need a bound on the depth of C_s in terms of the Ψ-weight of $\chi^{-1}(s)$.

For a color i, let $excess(i) := \Psi X(i) - \|\Psi\|_{avg}$ be the amount by which the Ψ-weight of $tent(i)$ exceeded the average Ψ-weight at the time when color i was pending. Similar to relation (8), $excess(i)$ decreases geometrically along the arcs (i,x) of $\mathcal{F}$. Since Lemma 8 gives stricter estimates for $\Phi^{(1)} = \Psi$, the claim below follows in fact from identities (6) and (7).

Claim 4. $excess(x) \leq \frac{1}{2}excess(i) + 2^{r-2}\|\Psi\|_\infty$

Proof: Identity (6) gives $\Psi X(x) = \Psi\chi^{-1}(x) + \Psi V_{in}(x)$. It holds $\Psi\chi^{-1}(x) \leq \|\Psi\|_{avg}$, because color x was *Light* in the beginning. Since (i,x) is a arc in $\mathcal{F}$, we get $\Psi V_{in}(x) \leq \Psi V_{out}(i)/2 + 2^{r-2}\|\Psi\|_\infty$ from Lemma 8 and identity (7). Again by identity (6) we have $\Psi V_{out}(i) = \Psi X(i) - \Psi\hat{\chi}^{-1}(i)$. By Claim 2 it holds $\Psi\hat{\chi}^{-1}(i) \geq \|\Psi\|_{avg}$ and thus $\Psi V_{out}(i) \leq excess(i)$. Now it follows

$$\begin{aligned} excess(x) &= \Psi V_{in}(x) + \Psi\chi^{-1}(x) - \|\Psi\|_{avg} \\ &\leq \Psi V_{out}(i)/2 + 2^{r-2}\|\Psi\|_\infty \\ &\leq \Psi(excess(i)/2 + 2^{r-2}\|\Psi\|_\infty. \end{aligned}$$

■

By the definition of the color set *Heavy*, every colors x with $excess(x) \le 2\|\Psi\|_{avg} + 2^r\|\Psi\|_\infty$ is either *Light* or *Medium*, and hence x is a leaf in $\mathcal{F}$. So it follows from Claim 4 and a simple inductive argument, that the depth of component C_s is at most logarithmic in the ratio of $\Psi\chi^{-1}(s)$ and $\|\Psi\|_{avg}$.

Claim 5. The depth d_s of a $\mathcal{F}$-component with root s is at most $\log(\Psi\chi^{-1}(s)/\|\Psi\|_{avg})$.

Proof: From Claim 4 it follows by induction on the depth d of node $x \in C_s$, that the following relation holds:

$$excess(x) \le 2^{-d} excess(s) + 2^{r-1}\|\Psi\|_\infty \le 2^{-d}\Psi\chi^{-1}(s) + 2^r\|\Psi\|_\infty$$

At depth $d_s - 1$ it must hold $\Psi\chi^{-1}(s)/2^{d_s-1} + 2^r\|\Psi\|_\infty \ge excess(x) > 2\|\Psi\|_{avg} + 2^r\|\Psi\|_\infty$ for at least one $x \in C_s$. It follows $\Psi\chi^{-1}(s) \ge 2^{d_s}\|\Psi\|_{avg}$ as claimed. ■

Claim 5 implies that the running time of our algorithm is $O(t(|G|) \cdot \log k)$. By Invariant 1, the vertex sets $X(i)$ are pairwise disjoint for all nodes i in the same level of $\mathcal{F}$, i.e., with the same distance from a root in $\mathcal{F}$. By linearity of t, the total time for the colors in one level is $O(t(|G|))$. So the total running time is $O(t(|G|) \cdot \log k)$ by Claim 5.

Since $\mathcal{F}[C_s]$ is a binary tree and the sets $X(i)$ are pairwise disjoint for nodes in the same level of $\mathcal{F}$, standard convexity arguments (Hölder's inequality) yield the claim below.

Claim 6. $\sum_{i\in C_s}\|c_{|X(i)}\|_p \le \sum_{d=0}^{d_s}\|c_{|A_s}\|_p \cdot 2^{d/q} \le 3q\|c_{|A_s}\|_p \cdot 2^{d_s/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $A_s := \bigcup_{i\in C_s} X(i)$.

Proof: Let L_s^d be the nodes in C_s with distance d from s. As $|L_s^d| \le 2^d$, we have by Hölder's inequality

$$\sum_{i\in L_s^d}\|c_{|X(i)}\|_p \le \|\sum_{i\in L_s^d} c_{|X(i)}\|_p \cdot 2^{d/q}.$$

Since the sets $X(i) \subseteq A_s$ are pairwise disjoint for $i \in L_s^d$, it holds $\sum_{i\in L_s^d} c_{|X(i)} \le c_{|A_s}$ and thus $\sum_{i\in L_s^d}\|c_{|X(i)}\|_p \le \|c_{|A_s}\|_p \cdot 2^{d/q}$. So we can conclude

$$\sum_{i\in C_s}\|c_{|X(i)}\|_p \le \sum_{0\le d\le d_s}\|c_{|A_s}\|_p \cdot 2^{d/q} \le \|c_{|A_s}\|_p \cdot \tfrac{2^{1/q}}{2^{1/q}-1} \cdot 2^{d_s/q} \le 3q\|c_{|A_s}\|_p \cdot 2^{d_s/q}$$

using $1/(2^{1/q} - 1) \ge q/\ln 2$ and $2^{1/q}/\ln 2 \le 3$. ■

Using the bound on d_s from Claim 5, and the bound on $\sum_{i\in C_s}\|c_{|X(i)}\|_p$ from Claim 6, we arrive with Hölder's inequality at:

Claim 7. $\sum_{i=1}^{k}\|c_{|X(i)}\|_p \le 3q\|c\|_p \cdot k^{1/q} = 3k \cdot \mathcal{B}$.

Proof: By Claim 5 we have $d_s \le \Psi\chi^{-1}(s)/\|\Psi\|_{avg} \le \Psi(A_s)/\|\Psi\|_{avg}$. Now it follows from Claim 6 and Hölder's Inequality,

$$\sum_{i=1}^{k}\|c_{|X(i)}\|_p \le \sum_s 3q\|c_{|A_s}\|_p(\tfrac{\Psi(A_s)}{\|\Psi\|_{avg}})^{1/q} \le 3q\Big\|\sum\nolimits_s c_{|A_s}\Big\|_p \cdot \Big(\sum\nolimits_s \tfrac{\Psi(A_s)}{\|\Psi\|_{avg}}\Big)^{1/q}.$$

Since $\{A_s\}_s$ form a partition of the vertex set V, we conclude

$$\sum_{i=1}^{k}\|c_{|X(i)}\|_p \le 3q\|c\|_p \cdot (\|\Psi\|_1/\|\Psi\|_{avg})^{1/q} = 3q\|c\|_p \cdot k^{1/q}.$$ ■

So by relation (9), $\|\partial\hat{\chi}^{-1}\|_{avg}$ is at most the average boundary cost of the original coloring χ plus $O_r(q \cdot \sigma_p \cdot \|c\|_p/k^{1/p})$. □

In the remainder of the section, we prove Proposition 7, which is our strongest result about multi-balanced colorings. The idea is to start with a coloring as obtained by Lemma 6, and then to balance the boundary costs of the coloring using essentially the algorithm of Lemma 9. The hope is that boundary costs of the color classes behave in the algorithm approximately like vertex measures. So, we should ensure that a single Move-step does not break the bound of Proposition 7 on the maximum boundary costs. For this reason, we balance the coloring beforehand with respect to the following measure.

Definition 10 (Splitting Cost Measure). The *(p-)splitting cost measure* of graph $G = (V, E)$ with respect to edge costs c is defined by

$$\pi \colon V \to \mathbb{R}_+, \quad \pi(v) := \sigma_p^p \sum_{e \in \delta(v)} c_e^p / 2.$$

For all vertex sets $W \subseteq V$, one has $\sigma_p \|c_{|W}\|_p \leq (\pi(W))^{1/p} =: \pi^{1/p}(W)$. So there exist splitting sets in $G[W]$ of boundary cost at most $\pi^{1/p}(W)$, even if weights and splitting value are worst possible. Thus, we call $\pi^{1/p}(W)$ the *splitting cost* of W.

Since $\|\pi\|_1^{1/p} = \sigma_p \|c\|_p$ and $\|\pi\|_\infty^{1/p} \leq \sigma_p \Delta_c$, we have

$$(\|\pi\|_1 / k + \|\pi\|_\infty)^{1/p} \leq \sigma_p (q \cdot k^{-1/p} \cdot \|c\|_p + \Delta_c) =: \mathcal{B}' \tag{10}$$

and hence color classes of π-balanced colorings can be split at cost $O(B')$. So if we start with a π-balanced coloring and maintain the π-balancedness, then a single iteration of procedure Move from Lemma 9 cannot break the bound on the maximum boundary cost from Proposition 7.

Proof of Proposition 7. We may assume that only the measures $\Phi^{(3)}$ through $\Phi^{(r)}$ are arbitrary and that the measures $\Phi^{(1)}$ and $\Phi^{(2)}$ can be defined by ourselves. This assumption does not weaken the statement of the proposition since the statement is invariant under renaming and adding (constantly many) "new" measures.

We define $\Phi^{(2)}$ to be the p-splitting cost measure π of G with respect to c. Let χ be a coloring that is balanced with respect to $\Phi^{(2)}$ through $\Phi^{(r)}$ and has average boundary cost at most proportional to $\sigma_p \cdot k^{-1/p} \cdot \|c\|_p$. By Lemma 6 such a coloring exists and can be obtained efficiently.

Consider the following measure that accounts for edges not running within a single color class of χ

$$\Psi(v) := c(\{uv \in E \mid \chi(u) \neq \chi(v)\}).$$

Note that $\|\partial\chi^{-1}\|_\infty = \|\Psi\chi^{-1}\|_\infty$, $\|\Psi\|_{avg} = \|\partial\chi^{-1}\|_{avg}$, and $\|\Psi\|_\infty \leq \Delta_c$. So if χ was Ψ-balanced, then the maximum boundary cost of χ would already be as required by the proposition.

We use Lemma 9 to establish Ψ-balancedness. However, there is a small twist. Instead of instantiating the lemma for r measures, we instantiate it for $r + 1$ measures, namely $\Phi^{(1)}, \ldots, \Phi^{(r+1)}$, where $\Phi^{(1)} := \Psi$, $\Phi^{(2)} := \pi$, and $\Phi^{(r+1)}$ is defined later. Then, let $\hat{\chi}$ be the coloring obtained by the algorithm of Lemma 9 for measures $\Phi^{(1)}$ through $\Phi^{(r+1)}$. So $\hat{\chi}$ is balanced with respect to $\Phi^{(1)}$ through $\Phi^{(r)}$ and has average boundary cost at most proportional to $\sigma_r \cdot k^{-1/p} \cdot \|c\|_p$.

The idea is now that for vertex sets $U \subseteq V$, the $\Phi^{(1)}$-weight of U approximates the boundary cost of U in the graph induced by the χ-bichromatic edges, and the $\Phi^{(r+1)}$-weight of U shall approximate the boundary cost of U within the monochromatic edges. So taking these two measure together should yield an approximation of the "real" boundary cost of U.

For the following proof, we define all symbols as in the proof of Lemma 6. In addition, let $E' := \{e \in E \mid |\chi(e)| = 1\}$ be the set of edges that run within one color class of χ. Note that the measure Ψ accounts for all edges besides E'. More specifically, we have for all $U \subseteq V$,

$$\partial U \leq \Psi U + \partial'(U) \tag{11}$$

where $\partial' U := c(\delta(U) \cap E')$ is the cost of the χ-monochromatic boundary edges.

Since $\hat{\chi}$ is balanced with respect to Ψ, we know $\|\Psi\hat{\chi}^{-1}\|_\infty = O_r(\|\Psi\|_{avg} + \|\Psi\|_\infty) = O_r(\mathcal{B}')$, Hence the estimate $\|\partial'\hat{\chi}^{-1}\|_\infty = O_r(\mathcal{B}')$ will imply the desired bound $\|\partial\hat{\chi}^{-1}\|_\infty \leq \|\Psi\hat{\chi}^{-1}\|_\infty + \|\partial'\hat{\chi}^{-1}\|_\infty = O_r(\mathcal{B}')$.

For deriving $\|\partial'\hat{\chi}^{-1}\|_\infty = O_r(\mathcal{B}')$, we need the following two technical claims.

Stronger than the π-balancedness of $\hat{\chi}$, the claim below holds – indeed implied by the discussion for Claim 3.

Claim 8. $\pi^{1/p}(X(i)) \ll_r (\|\pi\|_{avg} + \|\pi\|_\infty)^{1/p} \overset{(10)}{=} O_r(\mathcal{B}')$.

Proof: By Claim 3, we have $\pi(X(i)) \leq 4\|\pi\chi^{-1}\|_\infty + O_r(\|\pi\|_\infty)$. The balancedness of χ with respect to π implies $\|\pi\chi^{-1}\|_\infty = O_r(\|\pi\|_{avg} + \|\pi\|_\infty)$. ■

The next claim is a consequence of identity (6), and the fact that the edges between parts $\hat{\chi}^{-1}(i)$ and $V_{out}(i)$ have cost at most $\pi^{1/p}(X(i)) = O_r(\mathcal{B}')$. (Note that we can assume $\hat{\chi}^{-1}(i) \neq \chi^{-1}(i)$, since otherwise $\partial'\hat{\chi}^{-1}(i) = \partial'\chi^{-1}(i) = 0$.)

Claim 9. If $\hat{\chi}^{-1}(i) \neq \chi^{-1}(i)$ then $\partial'\hat{\chi}^{-1}(i) \leq \partial' V_{in}(i) + O_r(\mathcal{B}')$.

Proof: If the i-th color classes in χ and $\hat{\chi}$ differ, then $V_{in}(i)$ was defined at some point of the algorithm. Since $\hat{\chi}^{-1}(i) \subseteq X(i) = \chi^{-1}(i) \cup V_{in}(i)$ by identity (6), every boundary edge of $\hat{\chi}^{-1}(i)$ either crosses $X(i)$ or completely runs within $X(i)$. Thus $\partial'\hat{\chi}^{-1}(i) \leq \partial_{X(i)}\hat{\chi}^{-1}(i) + \partial' X(i)$. By construction, the boundary cost of $\hat{\chi}^{-1}(i)$ in $G[X(i)]$ is at most $\pi^{1/p}(X(i)) = O_r(\mathcal{B}')$. Since ∂' vanishes on $\chi^{-1}(i)$ as all its boundary edges are bichromatic in χ, we have $\partial' X(i) \leq \partial'\chi^{-1}(i) + \partial' V_{in}(i) = \partial' V_{in}(i)$. ■

For the bound $\|\partial'\hat{\chi}^{-1}\|_\infty = O_r(\mathcal{B}')$, it is now sufficient to show $\partial' V_{in}(i) = O_r(\mathcal{B}')$. The idea is to achieve a situation for $\partial' V_{in}(i)$ similar to relation (8), i.e., we want $\partial' V_{in}(i)$ to decrease geometrically along the arcs of $\mathcal{F}$. In fact, nothing else remains to show:

Claim 10. If $\partial' V_{in}(i)$ decreases geometrically along the arcs of $\mathcal{F}$, then the maximum boundary cost of $\hat{\chi}$ with respect to both ∂' and ∂ is in $O_r(\mathcal{B}')$.

Proof: If it holds $\partial' V_{in}(x) \leq z \cdot \partial' V_{in}(i) + O_r(\mathcal{B}')$ for some fixed constant $z < 1$ and all arcs (i, x) of $\mathcal{F}$, then we have $\partial' V_{in}(i) = O_r(\mathcal{B}')$ and by Claim 9 also $\partial'\hat{\chi}^{-1}(i) = O_r(\mathcal{B}')$ for all colors $i \in [k]$. Then, it follows from relation (11) that the maximum boundary cost of $\hat{\chi}^{-1}$ bounded by $\|\Psi\hat{\chi}^{-1}\|_\infty + O_r(\mathcal{B}')$. By construction, $\hat{\chi}^{-1}$ is balanced with respect to Ψ and thus $\|\Psi\hat{\chi}^{-1}\|_\infty \ll_r \|\Psi\|_{avg} + \|\Psi\|_\infty \ll_r \|\partial\chi^{-1}\|_\infty + \Delta_c = O_r(\mathcal{B}')$. Hence we get $\|\partial\hat{\chi}^{-1}\|_\infty = O_r(\mathcal{B}')$. ■

We choose the measure $\Phi^{(r+1)}$ in the following way to ensure the assumption of Claim 10. At the time when Move is applied to color $i \in [k]$, let $\Phi^{(r+1)}(v)$ be the cost of the edges from $\delta(V_{in}(i)) \cap E'$ that are incident to $v \in V$ Formally, $\Phi^{(r+1)}(v) := c(\delta(v) \cap \delta(V_{in}(i)) \cap E')$ for $v \in V_{in}(i)$. For convenience, we set $\Phi^{(r+1)}(v) = 0$ for vertices outside of $V_{in}(i)$. We have $\Phi^{(r+1)}V_{in}(i) = \partial' V_{in}(i)$ and for all $U \subseteq V$, in particular $U = V_{in}(i)$, it holds

$$\partial' U \leq \Phi^{(r+1)}(U) + \partial_{X(i)} U \tag{12}$$

by identity (6) and the fact $\partial' \chi^{-1}(i) = 0$. So $\Phi^{(r+1)}$ is a good approximation of ∂', at least for sets with small boundary cost in $G[X(i)]$.

For an arc (i, x) of $\mathcal{F}$, it follows from Claim 8 and Lemma 8 that the boundary cost of $V_{in}(x)$ in $G[X(i)]$ is at most proportional to $\pi^{1/p}(X(i)) = O_r(\mathcal{B}')$. Lemma 8 guarantees $\Phi^{(r+1)}V_{in}(x) \leq 3/4 \cdot \Phi^{(r+1)}V_{out}(i) + O_r(\|\Phi^{(r+1)}\|_\infty)$. Then relation (12) and the fact $\Phi^{(r+1)}V_{out}(i) \leq \partial' V_{in}(i)$ finally show:

Claim 11. $\partial' V_{in}(x) \leq \frac{3}{4}\partial' V_{in}(i) + O_r(\mathcal{B}')$.

Proof: Using $\partial_{X(i)} V_{in}(x) = O_r(\mathcal{B}')$ and $\Phi^{(r+1)}V_{in}(x) \leq 3/4 \cdot \Phi^{(r+1)}V_{out}(i) + O_r(\|\Phi^{(r+1)}\|_\infty)$, we get from relation (12): $\partial' V_{in}(x) \leq 3/4 \cdot \Phi^{(r+1)}V_{out}(i) + O_r(\mathcal{B}' + \|\Phi^{(r+1)}\|_\infty)$. Clearly, $\|\Phi^{(r+1)}\|_\infty \leq \mathcal{B}'$. Since $\|\Phi^{(r+1)}\|_1 = \partial' V_{in}(i)$ when MOVE is applied to color i, we have $\Phi^{(r+1)}V_{out}(i) \leq \partial' V_{in}(i)$ and therefore $\partial' V_{in}(x) \leq 3/4 \cdot \partial' V_{in}(i) + O_r(\mathcal{B}')$, as required. ■

From Claim 10 it follows now that $\hat{\chi}$ fulfills all requirements of the proposition. □

4 Improving balancedness at no cost

In this section we show how weakly balanced colorings can be transformed into strictly balanced colorings while maintaining the bounds on the maximum boundary cost claimed by Theorem 4. Together with Proposition 7, this result shall imply Theorem 4.

We proceed in two steps. First we obtain a similar result about colorings with only slightly relaxed constraints on the weights. A k-coloring is called *almost strictly balanced* with respect to weights w if the weight of each color classes differs from the average weight by at most $2\|w\|_\infty$.

Proposition 11. *Let $k \in \mathbb{N}$ and $G = (V, E)$ be a graph with edge costs c and vertex weights w.*

Then any w-balanced k-coloring χ of G can be transformed into an almost strictly balanced k-coloring $\hat{\chi}$ without increasing the maximum boundary cost or splitting cost by more than a constant factor, essentially. More precisely,

$$\|\pi\hat{\chi}^{-1}\|_\infty = O_p(\|\pi\chi^{-1}\|_\infty)$$
$$\|\partial\hat{\chi}^{-1}\|_\infty = O_p(\|\partial\chi^{-1}\|_\infty + \|\pi\chi^{-1}\|_\infty^{1/p})$$

where π is the p-splitting cost measure of G (cf. Definition 10).

The coloring $\hat{\chi}$ can be obtained from χ in time $O(t(|G|))$ with t is as in Theorem 4.

As soon as we have an almost strictly balanced coloring its is easy to obtain a desired strictly balanced coloring. The idea is to reduce the weight of each color class below $\|w\|_{avg}$ by cutting off parts (vertex sets) of weight about $\|w\|_\infty$. These parts are then redistributed among the color classes by a greedy bin-packing procedure. Since we started with an almost strictly balanced coloring, the above procedure alters each color class at most a constant number of times. So we get the proposition below. For a detailed proof we refer to Appendix A.2.

Proposition 12. *Let k and G be as in Proposition 11. Then any almost balanced k-coloring χ can be turned into a strictly balanced k-coloring with*

$$\|\partial\hat{\chi}^{-1}\|_\infty = O_p(\|\partial\chi^{-1}\|_\infty + \|\pi\chi^{-1}\|_\infty^{1/p} + \Delta_c).$$

The coloring $\hat{\chi}$ can be obtained from χ in time $O(t(|G|)\log k)$, where t is as in Theorem 4.

Theorem 4 is implied by the conjunction of Propositions 7, 11, and 12.

Proof of Theorem 4. If we apply Proposition 7 with $\Phi^{(1)} := w$ and $\Phi^{(2)} := \pi$, then we obtain a w-balanced coloring χ_1 such that both the maximum splitting cost $\|\pi\chi_1^{-1}\|_\infty^{1/p}$ and the maximum boundary cost $\|\partial\chi_1^{-1}\|_\infty$ of χ_1 are at most proportional to $\sigma_p \cdot (k^{-1/p} \cdot \|c\|_p + \Delta_c)$.

By Proposition 11, we can transform χ_1 into an almost strictly w-balanced coloring χ_2 with maximum splitting cost and maximum boundary cost fulfilling the same bounds as before.

Finally, Proposition 12 yields a strictly w-balanced coloring χ_3 of G with maximum boundary cost $O_p(\sigma_p \cdot (k^{-1/p}\|c\|_p + \Delta_c)$. Hence it holds $\partial_\infty^k(G, c) = O_p(\sigma_p \cdot (k^{-1/p}\|c\|_p + \Delta_c)$. □

In the remainder of this section we sketch a proof of Proposition 11.

We aim for a recursive algorithm that computes an almost strictly balanced coloring from a weakly balanced k-coloring χ, with $\|w\chi^{-1}\|_\infty \leq M\|w\|_{avg}$ for some constant M. First, a so called *shrinking procedure* computes from χ two colorings $\chi_0 \colon V_0 \to [k]$ and $\chi_1 \colon V_1 \to [k]$ of disjoint vertex subsets V_0 and V_1 with $V_0 \cup V_1 = V$. The coloring χ_0 shall be almost strictly balanced and χ_1 is weakly balanced with $\|w\chi_1^{-1}\|_\infty \leq M\|w_{|V_1}\|_{avg}$. Most importantly, the maximum splitting cost and the maximum boundary cost decrease geometrically when going from coloring χ to the "shrunken" coloring χ_1.

From coloring χ_1, we recursively compute an almost strictly balanced coloring $\hat{\chi}_1$ of V_1. Since now both χ_0 and $\hat{\chi}_1$ are almost strictly balanced, the weight of each color class in the direct sum $\chi_0 \oplus \hat{\chi}_1 \colon V \to [k]$ differs from the average weight by at most $4\|w\|_\infty$. So coloring χ_0 need to be changed only slightly to obtain a coloring $\tilde{\chi}_0$ such that the direct sum $\tilde{\chi}_0 \oplus \hat{\chi}_1$ indeed is the desired almost strictly balanced coloring $\hat{\chi}$ of V.

In order to ensure that the boundary costs do not accumulate in the recursive calls, we need precise bounds on the maximum boundary cost and maximum splitting cost of the colorings χ_0 and χ_1. The following definition captures these (technical) requirements. Roughly speaking, one wants that the maximum boundary cost $\|\partial\chi_0^{-1}\|_\infty$ and the maximum splitting cost $\|\pi\chi_0^{-1}\|_\infty$ of coloring χ_0 are at most proportional to the respective costs of the original coloring χ. The costs $\|\partial\chi_1^{-1}\|_\infty$ and $\|\pi\chi_1^{-1}\|_\infty$ of the coloring χ_1 should be geometrically less than the respective costs in χ. In order to ensure that our recursive algorithm runs in linear time, we require that the size $|G[V_1]|$ of the graph induced by V_1 is only a constant fraction of $|G|$.

Definition 13 (Shrinking Procedure). For $\epsilon > 0$ and $M := 1/\epsilon^5$, let P be a procedure that transforms any weakly balanced k-coloring χ of a vertex set $W \subseteq V$ with

$$\|w\chi^{-1}\|_\infty \leq M \cdot \|w_{|W}\|_{avg} \quad \text{and} \quad \|w\|_\infty \leq \epsilon^5 \cdot \|w_{|W}\|_{avg}$$

into two k-colorings χ_0 and χ_1 of disjoint sets W_0 and W_1 with $W_0 \cup W_1 = W$.

Then procedure P is called *ϵ-shrinking* if

a) coloring χ_0 is almost strictly balanced with $w\chi_0^{-1}(i) - \epsilon\|w_{|W}\|_{avg} \in [0, \|w\|_\infty]$, and it holds $\|\pi\chi_0^{-1}\|_\infty = O_M(\|\pi\chi^{-1}\|_\infty)$, and also $\|\partial\chi_0^{-1}\|_\infty = O_M(\|\partial\chi^{-1}\|_\infty + \|\pi\chi^{-1}\|_\infty^{1/p})$,

b) coloring χ_1 is weakly balanced with $\|w\chi_1^{-1}\|_\infty \le M\|w_{|W_1}\|_{avg}$, and it holds $\|\pi\chi_1^{-1}\|_\infty \le (1-\epsilon^{10})\|\pi\chi^{-1}\|_\infty$, and also $\|\partial\chi_1^{-1}\|_\infty \le (1-\epsilon^{10})\|\partial\chi^{-1}\|_\infty + O_M(\|\pi\chi^{-1}\|_\infty^{1/p})$,

c) it holds $|G[W_1]| \le (1-\epsilon^{10})|G[W]|$.

Notice that $\partial\chi_b^{-1}$ and $\partial\chi^{-1}$ refer to the boundary costs of the respective color classes with respect to the host graph G (as opposed to $G[W_b]$ or $G[W]$).

The definition above makes only sense if there are efficient shrinking procedures.

Lemma 14. *For sufficiently small $\epsilon > 0$, there exist ϵ-shrinking procedures that run in time proportional to $t(|G[W]|)$ when applied to a balanced coloring of $G[W]$.*

We can think of our algorithm as a divide-and-conquer algorithm. Then the shrinking procedure divides the problem into two subproblems, where the subproblem corresponding to the coloring χ_0 is trivial, since χ_0 is already almost strictly balanced, and the subproblem for coloring χ_1 is of the same type as for the input coloring χ but has reduced complexity, in the sense that the maximum splitting and boundary costs decreased geometrically. We do not know yet how the conquer-phase works, i.e., how to construct a solution for the original problem.

Suppose we obtained an almost strictly balanced coloring $\hat{\chi}_1$ from the recursive call for χ_1. We want to transform the coloring χ_0 into a coloring $\tilde{\chi}_0$ such that the direct sum of $\tilde{\chi}_0$ and χ_1 is almost strictly balanced. The idea is to uncolor parts of the color classes in χ_0 until the direct sum with χ_1 has maximum weight at most $\|w\|_{avg}$. The weight of each part shall be between $\|w\|_\infty$ and $2\|w\|_\infty$. Then these parts are redistributed among the color classes by a greedy bin-packing procedure, so that the direct sum $\tilde{\chi}_0 \oplus \hat{\chi}_1$ is almost strictly balanced.

With the technical requirements of the lemma below, a straight-forward analysis shows that each color class is changed only constantly often in the conquer-phase and thus the the maximum splitting cost measure or the maximum boundary cost increased by no more than a constant factor (essentially).

Lemma 15 (Conquer-Phase). *Let χ_0 and $\hat{\chi}_1$ be two k-colorings of disjoint sets W_0 and W_1 with $W_0 \cup W_1 = W$. Suppose that $w\hat{\chi}_1^{-1}(i) \le \|w_{|W}\|_{avg} - \|w\|_\infty$ for each color $i \in [k]$.*

If both $w\chi_0^{-1}(i) = \|w_{|W_0}\|_{avg} + O(\|w\|_\infty)$ and $w\hat{\chi}_1^{-1}(i) = \|w_{|W_1}\|_{avg} + O(\|w\|_\infty)$ for all colors $i \in [k]$, then χ_0 can be transformed into a coloring $\tilde{\chi}_0$ such that the direct sum $\hat{\chi} = \tilde{\chi}_0 \oplus \hat{\chi}_1$ is almost strictly balanced.

Neither the maximum splitting cost nor the maximum boundary cost is increased by more than essentially a constant factor; more precisely, $\|\pi\tilde{\chi}_0^{-1}\|_\infty = O(\|\pi\chi_0^{-1}\|_\infty)$ and $\|\partial\tilde{\chi}_0^{-1}\|_\infty = O(\|\partial\chi_0^{-1}\|_\infty + \|\pi\chi_0^{-1}\|_\infty^{1/p})$.

The coloring $\tilde{\chi}_0$ can be obtained in time $O(t(|G|))$.

Assuming Lemma 14 and Lemma 15, a straight-forward analysis of the described "shrink-and-conquer" algorithm proves Proposition 11. (Lemma 15 is also used to handle the base case of the algorithm, i.e., for $\|w\|_\infty \ge \epsilon^5\|w_{|W}\|_{avg}$ when the ϵ-shrinking procedure cannot be applied.)

For proofs of the assumed lemmas 14 and 15 we refer to Section 5 and Appendix A.2, respectively.

Proof of Proposition 11. We show by induction on the cardinality of W that the coloring χ can be transformed into an almost strictly balanced coloring $\hat{\chi}$ with $\|\pi\hat{\chi}^{-1}\|_\infty \leq C_1\|\pi\chi^{-1}\|_\infty$ and $\|\partial\hat{\chi}^{-1}\|_\infty \leq C_1\|\partial\chi^{-1}\|_\infty + C_2\|\pi\chi^{-1}\|_\infty^{1/p}$ for appropriately chosen constants C_1, C_2. Let P be an ϵ-shrinking procedure from Lemma 14 for some sufficiently small absolute constant $\epsilon > 0$. Notice that $M := 1/\epsilon^5$ is also an absolute constant.

The base case is $\|w\|_\infty > \epsilon^5\|w_{|W}\|_{avg}$. Since χ is weakly balanced, the maximum weight of χ is at most $M \cdot \|w_{|W}\|_{avg} \leq M^2\|w\|_\infty$. Hence we can apply Lemma 15 (with $W_0 = W$ and $W_1 = \emptyset$) to obtain an almost strictly balanced coloring $\tilde{\chi}_0$. The coloring $\hat{\chi} := \tilde{\chi}_0$ satisfies the requirements of the proposition.

We may now assume $\|w\|_\infty \leq \epsilon^5\|w_{|W}\|_{avg}$ and $\|w_{|W}\|_{avg} > 0$. Then we can apply our ϵ-shrinking procedure P to obtain colorings χ_0 and χ_1 of disjoint vertex sets W_0 and W_1. By induction hypothesis, χ_1 can be transformed into an almost strictly balanced coloring $\hat{\chi}_1$ with $\|\pi\hat{\chi}_1^{-1}\|_\infty \leq C_1\|\pi\chi_1^{-1}\|_\infty$ and $\|\partial\hat{\chi}_1^{-1}\|_\infty \leq C_1\|\partial\chi_1^{-1}\|_\infty + C_2\|\pi\chi_1^{-1}\|_\infty^{1/p}$.

The maximum weight $\|w\hat{\chi}_1^{-1}\|_\infty$ is bounded by

$$\|w_{|W_1}\|_{avg} + 2\|w\|_\infty \leq (1 - \epsilon + 3\epsilon^5)\|w_{|W}\|_{avg} - \|w\|_\infty$$

which is less than the upper bound $\|w_{|W}\|_{avg} - \|w\|_\infty$ required by Lemma 15, for sufficiently small ϵ. Thus, we can apply Lemma 15 to obtain a coloring $\tilde{\chi}_0$ such that $\tilde{\chi}_0 \oplus \hat{\chi}_1$ is almost strictly balanced.

The maximum π-weight of $\hat{\chi} := \tilde{\chi}_0 \oplus \hat{\chi}_1$ is at most $\|\pi\hat{\chi}_1^{-1}\|_\infty + \|\pi\tilde{\chi}_0^{-1}\|_\infty \leq C_1\|\pi\chi_1^{-1}\|_\infty + O(\|\pi\chi^{-1}\|_\infty)$. Since P is ϵ-shrinking, it holds $\|\pi\chi_1^{-1}\|_\infty \leq z\|\pi\chi^{-1}\|_\infty$ for $z := (1 - \epsilon^{10})$ and therefore we have

$$\|\pi\hat{\chi}^{-1}\|_\infty \leq (z \cdot C_1 + O(1))\|\pi\chi^{-1}\|_\infty \leq C_1\|\pi\chi^{-1}\|_\infty$$

for sufficiently large C_1.

Similarly, the maximum boundary cost of $\hat{\chi}$ is at most $C_1 \cdot (z\|\partial\chi^{-1}\|_\infty + O(\|\pi\chi^{-1}\|_\infty^{1/p}) + C_2 \cdot (z^{1/p} \cdot \|\pi\chi^{-1}\|_\infty) + O(\|\partial\chi^{-1}\|_\infty + \|\pi\chi^{-1}\|_\infty^{1/p})$. It holds

$$\begin{aligned}\|\partial\hat{\chi}^{-1}\|_\infty &\leq (z \cdot C_1 + O(1))\|\partial\chi^{-1}\|_\infty \\ &\quad + (z^{1/p} \cdot C_2 + O(C_1 + 1))\|\pi\chi^{-1}\|_\infty^{1/p} \\ &\leq C_1\|\partial\chi^{-1}\|_\infty + (z^{1/p}C_2 + O(C_1))\|\pi\chi^{-1}\|_\infty^{1/p}\end{aligned}$$

for sufficiently large C_1. Then if C_2 is large enough relative to C_1, the maximum boundary cost of $\hat{\chi}$ is at most $C_1 \cdot \|\partial\chi^{-1}\|_\infty + C_2 \cdot \|\pi\chi^{-1}\|_\infty^{1/p}$.

The claimed running time $O(t(|G|))$ follows from the facts that t is a linear function and that the size of the considered graph decreases by a constant factor with each application of the shrinking procedure. □

5 Shrinking procedure

In this section we show Lemma 14. Let $\epsilon > 0$ be a sufficiently small absolute constant. The precise value of ϵ is not important. For convenience, we write $M := \epsilon^5$, $\Psi := w$, $\Phi^{(1)} := \pi$, and $\Phi^{(2)} := \deg_W$, where $\deg_W(v)$ is the degree of v in $G[W]$. Notice that $\Phi^{(2)}(W_1) \leq (1-\epsilon^{10})\Phi^{(2)}(W)$ implies $|G[W_1]| \leq (1-\epsilon^{10})|G[W]|$ for all vertex sets $W_1 \subseteq W$.

In the following we assume $\|\Psi\|_\infty \leq \epsilon^5\Psi^*$, where $\Psi^* := \Psi(W)/k$ is the average weight of a color class in coloring $\chi : W \to [k]$. We need the corollaries below for our proof of Lemma 14. (For the proof and lemma of the corollaries we refer to Appendix A.1.)

A vertex set U with Ψ-weight $\Theta(M \cdot \Psi^*)$ can be partitioned into about $\Theta(M/\epsilon)$ parts, each of Ψ-weight $\Theta(\epsilon\Psi^*)$. An averaging argument shows that for one of these parts, X say, all three $\Phi^{(1)}(X)$, $\Phi^{(2)}(X)$, and $\partial(X)$ are small – at most an $O(\epsilon/M)$-fraction:

Corollary 16. *For every $U \subseteq V$ with $M/2 \leq \Psi(U)/\Psi^* \leq M$, there exists a subset X of U with $\partial_U X = O_M(\pi^{1/p}(U))$ and $\epsilon \leq \Psi(X)/\Psi^* \leq 3\epsilon$ such that*

$$\begin{aligned}\Phi^{(j)}(X) &\leq (18\epsilon/M) \cdot \Phi^{(j)}(U)\\ \partial X &\leq (18\epsilon/M) \cdot \partial U + O_M(\pi^{1/p}(U))\end{aligned}$$

Analogous to the corollary above:

Corollary 17. *For every $U \subseteq V$ with $1/2 \leq \Psi(U)/\Psi^* \leq M$, there exists a subset X of U with $\partial_U X = O_M(\pi^{1/p}(U))$ and $\epsilon \leq \Psi(X)/\Psi^* \leq 3\epsilon$ such that*

$$\begin{aligned}\Phi^{(j)}(X) &\leq 18\epsilon \cdot \Phi^{(j)}(U),\\ \partial X &\leq 18\epsilon \cdot \partial U + O_M(\pi^{1/p}(U))\end{aligned}$$

Somehow dual to the preceding corollaries. A vertex set U is partitioned into at most $9\Psi(U)/(\epsilon \cdot \Psi^*)$ parts, each of Ψ-weight about $\epsilon/9 \cdot \Psi^*$. Among these parts, let X_1, X_2, X_3 be the parts with maximum $\Phi^{(1)}$-, $\Phi^{(2)}$, and ∂-weight, respectively. Then for the union $X = X_1 \cup X_2 \cup X_3$, all three $\Phi^{(1)}(X)$, $\Phi^{(2)}(X)$ and $\partial(X)$ are large – at least a $(\epsilon/9 \cdot \Psi^*/\Psi(U))$-fraction.

Corollary 18. *For every $U \subseteq V$ with $\epsilon \leq \Psi(U)/\Psi^* \leq M$, there exists a subset X of U with $\partial_U X = O_M(\pi^{1/p}(U))$ and $\epsilon \leq \Psi(X)/\Psi^* \leq \epsilon + \|\Psi\|_\infty/\Psi^*$ such that*

$$\begin{aligned}\Phi^{(j)}(U\backslash X) &\leq (1 - \epsilon/9 \cdot \tfrac{\Psi^*}{\Psi(U)}) \cdot \Phi^{(j)}(U),\\ \partial(U\backslash X) &\leq (1 - \epsilon/9 \cdot \tfrac{\Psi^*}{\Psi(U)}) \cdot \partial U + O_M(\pi^{1/p}(U))\end{aligned}$$

Now we are armed to show Lemma 14. We remark that sets X as in the corollaries above can be obtained in time $O_M(t(|G[U]|))$.

Proof of Lemma 14. We are given a coloring χ of a vertex set $W \subseteq V$ with $\|\Psi\chi^{-1}\|_\infty \leq M\Psi^*$ Our aim is to find two k-coloring χ_0 and χ_1 with each vertex of W being colored in exactly one of two colorings, such that χ_0 is almost strictly Ψ-balanced and χ_1 is weakly balanced with $\|\Psi\chi_1^{-1}\|_\infty \leq M \cdot \|\Psi\chi_1^{-1}\|_{avg}$.

First, we transform coloring χ into a coloring $\tilde{\chi}$ with maximum Ψ-weight at most $M/2 \cdot \Psi^*$ and minimum Ψ-weight at least $\epsilon \cdot \Psi^*$. This transformation is done by moving parts generated by Corollary 16 and 17 from "heavy" color classes to "light" color classes. Then Corollary 18 can be applied to each color class $\tilde{\chi}^{-1}(i)$. The corollary yields sets $X_i \subseteq \tilde{\chi}^{-1}(i)$ with $\Psi(X_i)$ at least $\epsilon\Psi^*$ and at most this value plus $\|\Psi\|_\infty$. Hence, the restriction of $\tilde{\chi}$ to the union W_0 of the sets X_1 to X_k is an almost strictly balanced coloring. So we can define $\chi_0 := \tilde{\chi}_{|W_0}$. On the other hand, the restriction of $\tilde{\chi}$ to the complement $W_1 := W \setminus W_0$ is a coloring with maximum Ψ-weight at most $M/2 \cdot \Psi^*$. It is not difficult to check that $M/2 \cdot \Psi^* \leq M \cdot \Psi(W_1)/k$ (cf. Claim 1). So coloring $\chi_1 := \tilde{\chi}_{|W_1}$ fulfills $\|\Psi\tilde{\chi}_1^{-1}\|_\infty \leq M \cdot \|\Psi\chi_1^{-1}\|_{avg}$.

We need to ensure that the construction of $\tilde{\chi}$ does not increase the $\Phi^{(j)}$-weight of color classes by more than a small fraction of $\|\Phi^{(j)}\chi^{-1}\|_\infty$. Otherwise, Corollary 18 could not guarantee $\|\Phi^{(j)}\chi_1^{-1}\|_\infty \leq (1-\epsilon^{10})\|\Phi^{(j)}\chi^{-1}\|_\infty$ as required by the lemma. Similarly, the construction should not increase the boundary costs of color classes by too much.

The idea for constructing $\tilde{\chi}$ is as follows. We start with $\tilde{\chi} = \chi$. First the maximum Ψ-weight of $\tilde{\chi}$ is reduced to $M/2 \cdot \Psi^*$. For color classes $\tilde{\chi}^{-1}(i)$ with Ψ-weight larger than $M/2 \cdot \Psi^*$, we uncolor subsets $X \subseteq \tilde{\chi}^{-1}(i)$ as in Corollary 16 and store these sets in a data structure called *Buffer* $\subseteq 2^W$. The corollary ensures that all parts $X \in$ *Buffer* have Ψ-weight about $\epsilon\Psi^*$ but only very small $\Phi^{(j)}$-weight. Then the minimum Ψ-weight of $\tilde{\chi}$ is increased to $\epsilon\Psi^*$. If *Buffer* had enough elements, we could simply assign one part $X \in$ *Buffer* to each color class with Ψ-weight less than $\epsilon\Psi^*$. Otherwise, we have to use Corollary 17 to generate more parts (from color classes with Ψ-weight at least $\Psi^*/2$). Note that the parts generated by this corollary are M times more costly than the parts generated by Corollary 16. In case that *Buffer* contained more elements than there were color classes with weight below $\epsilon\Psi^*$, we distribute the remaining parts of *Buffer* greedily among the color classes. An important observation about our construction that if we assigned more than one part X to a color class, then all theses parts are as in Corollary 16.

It remains to give a detailed description of the shrinking procedure. As indicated before, we start with $\tilde{\chi} = \chi$. For convenience, we subdivide the procedure into three phases (subroutines) CutDown, AddTo and ReduceBuffer.

The procedure CutDown is used to reduce the Ψ-weight of a color class by a constant fraction of Ψ^*. The algorithm will iterate this procedure until each color class of $\tilde{\chi}$ has weight at most $M/2 \cdot \Psi^*$.

Procedure CutDown (color $i \in [k]$)

// *Precondition:* $M/2 \cdot \Psi^* < \Psi\,\tilde{\chi}^{-1}(i) \leq M \cdot \Psi^*$

(1.) Compute a subset X of $\tilde{\chi}^{-1}(i)$ with $\epsilon \leq \Psi(X)/\Psi^* \leq 3\epsilon$ as in Corollary 16

(2.) Uncolor all vertices of X in $\tilde{\chi}$

(3.) Insert set X into *Buffer*

The procedure AddTo increases the Ψ-weight of color class by assigning a part $X \subseteq W$ to it. Either part X is an element of *Buffer*, or X is a subset of some color class $\tilde{\chi}^{-1}(j)$ as in Corollary 17. When procedure AddTo is iterated appropriately, we yield a coloring $\tilde{\chi}$ with each color class having Ψ-weight at least $\epsilon\Psi^*$.

Procedure AddTo (color $j \in [k]$)

// *Precondition:* $\Psi\,\tilde{\chi}^{-1}(j) < \epsilon \cdot \Psi^*$

(1.) If *Buffer* $= \emptyset$,

then let i be a color with $\Psi\,\tilde{\chi}^{-1}(i) \geq \Psi^*/2$,

compute a subset X of $\tilde{\chi}^{-1}(i)$ with

$\epsilon \leq \Psi(X)/\Psi^* \leq 3\epsilon$ as in Corollary 17,

else

let X be an arbitrary element of *Buffer*,

and remove X from *Buffer*

(2.) Paint all vertices in X with color j.

Finally, the procedure ReduceBuffer is used to empty the buffer in case that *Buffer* contained more elements than there were color classes with weight below $\epsilon\Psi^*$. A part $X \in$ *Buffer* is simply assigned to a color class with at most average Ψ-weight.

Procedure ReduceBuffer ()

// *Precondition: Buffer* $\neq \emptyset$

(1.) Remove some part X from the *Buffer*.

(2.) Let $j \in [k]$ be a color with $\Psi\,\tilde{\chi}^{-1}(j) \leq \Psi^*$

(3.) Paint all vertices in X with color j.

Now our shrinking procedure reads as follows.

Procedure Shrink (coloring $\chi\colon W \to [k]$)

// *Precondition:* $\|\Psi\chi^{-1}\|_\infty \leq M\Psi^*$

(1.) Start with $\tilde{\chi} \leftarrow \chi$, and *Buffer* $\leftarrow \emptyset \subseteq 2^W$.

(2.) As long as $\exists$ a color i with $\Psi\,\tilde{\chi}^{-1}(i) > M/2 \cdot \Psi^*$,
do CutDown (i).

(3.) For every color i with $\Psi\,\tilde{\chi}^{-1}(i) < \epsilon\Psi^*$,
do AddTo (i).

(4.) Until *Buffer* $= \emptyset$,
do ReduceBuffer ().

// *Assert:* $\tilde{\chi}$ *is a total coloring of* W *with* $\epsilon\Psi^* \leq \Psi\tilde{\chi}^{-1}(i) \leq M/2 \cdot \Psi^*$ *for all* $i \in [k]$.

(5.) For each color $i \in [k]$,
compute a subset X_i of $\tilde{\chi}^{-1}(i)$ as in Cor. 18
with $\epsilon\Psi^* \leq \Psi(X_i) \leq \epsilon\Psi^* + \|\Psi\|_\infty$.

(6.) Set $W_0 := X_1 \cup \ldots \cup X_k$ and $W_1 := W \setminus W_0$.

(7.) Return the colorings $\chi_0 := \tilde{\chi}_{|W_0}$ and $\chi_1 := \tilde{\chi}_{|W_1}$.

The assertion before step (5.), which is easily seen to hold, and the fact $\Psi(X_i) - \epsilon\Psi^* \in [0, \|\Psi\|_\infty] \subseteq [0, \epsilon^5\Psi^*]$ imply the claim below (for sufficiently small ϵ).

Claim 1. If χ_0 and χ_1 are the colorings computed by Shrink (χ), then χ_0 is almost strictly balanced with $\epsilon\Psi^* \leq \Psi\chi_0^{-1}(i) = \Psi(X_i) \leq \epsilon\Psi^* + \|\Psi\|_\infty$ and χ_1 satisfies $\|\Psi\chi_1^{-1}\|_\infty \leq M \cdot \Psi_1^*$, where $\Psi_1^* := \Psi(W_1)/k$ is the average Ψ-weight of a k-coloring of W_1.

Proof: By construction, coloring χ_0 fulfills the claim. To see that the claim holds for χ_1, we observe $\Psi(W_1)/k \geq \Psi(W)/k - \epsilon\Psi^* - \|\Psi\|_\infty$. So it holds $\Psi_1^* \geq (1 - \epsilon - \epsilon^5)\Psi^*$, and therefore $\|\Psi\chi_1^{-1}\|_\infty \leq M/2 \cdot \Psi^* \leq M\Psi_1^*$ for all sufficiently small $\epsilon > 0$. ■

The next claim observes a simple but crucial property of the algorithm. The colors are naturally divided into donators and receivers. Formally, let *Source* $\subseteq [k]$ be the set of colors i with $\chi^{-1}(i) \setminus \tilde{\chi}^{-1}(i) \neq \emptyset$, and *Sink* $\subseteq [k]$ be the set of colors j with $\tilde{\chi}^{-1}(j) \setminus \chi^{-1}(j) \neq \emptyset$.

Clearly, *Source* consists exactly of the colors for which we called procedure CutDown, or that were selected in step (1.) of AddTo. Similarly, *Sink* consists of all colors for which we called procedure AddTo, or that were selected in step (2.) of ReduceBuffer. For sufficiently small ϵ, no color can be both a source and a sink:

Claim 2. Source $\cap$ *Sink* $= \emptyset$

Proof: We distinguish two cases.

First, we consider the case that CutDown created more *Buffer*-parts than were used by AddTo – so ReduceBuffer was called at least once. Then *Source* consists only of those colors $i \in [k]$ for which CutDown(i) has been called. Hence for all colors $i \in \mathit{Source}$,

$$\Psi\tilde{\chi}^{-1}(i) \geq (M/2 - 3\epsilon)\Psi^* > \Psi^*$$

is an invariant of the algorithm. From this invariant it follows that *Source* contains no color $j \in [k]$ for which AddTo(j) was called or that was selected in step (2) of procedure ReduceBuffer. So, *Source* and *Sink* are disjoint sets.

In the case that ReduceBuffer was never called by the algorithm, *Sink* consists only of those colors $j \in [k]$ for which AddTo(j) was called. So for all colors $j \in \mathit{Sink}$,

$$\Psi\tilde{\chi}^{-1}(j) \leq (\epsilon + 3\epsilon)\Psi^* < \Psi^*/2$$

is an invariant of the algorithm. This invariant implies that *Source* cannot contain a color $i \in [k]$ for which CutDown(i) was called or that was selected in step (1) of procedure AddTo. Therefore, *Source* and *Sink* must be mutually exclusive. ■

Based on Claim 2, we show bounds on the $\Phi^{(j)}$-weight and boundary cost of the color classes in $\tilde{\chi}$. These bounds will imply that the maximum $\Phi^{(j)}$-measure and the maximum boundary cost of both χ_0 and χ_1 are as required by the lemma.

Consider any source color $i \in \mathit{Source}$. Every time when its color class is changed by the algorithm, the Ψ-weight of $\tilde{\chi}^{-1}(i)$ decreases by at least $\epsilon\Psi^*$. Since the initial weight is at most $M \cdot \Psi^*$, there can be at most M/ϵ such changes. Thus, the increase of $\partial\tilde{\chi}^{-1}(i)$ is bounded by $M/\epsilon \cdot O_M(\pi^{1/p}(\chi^{-1}(i))) = O_M(\|\pi\chi^{-1}\|_\infty^{1/p})$, because each applied cut has cost $O_M(\pi^{1/p}(\tilde{\chi}^{-1}(i)))$. This observation shall imply that for the analysis of our shrinking algorithm, the boundary cost function ∂ behaves like one of the vertex measures $\Phi^{(j)}$ modulo additive terms of order $\|\pi\chi^{-1}\|_\infty^{1/p}$. So we restrict ourselves in the following to show the requirements of the lemma only for $\Phi^{(j)}$. The arguments for ∂ are analogous.

Consider a non-sink color i, and let $U = \tilde{\chi}^{-1}(i)$ be its color class in step (5.) of procedure Shrink. By the assertion before step (5.), it holds $\Psi^*/\Psi U \geq 2/M$. Thus $\chi_1^{-1}(i) = U \setminus X_i$ has $\Phi^{(j)}$-weight at most $(1 - \frac{2\epsilon}{9M})\Phi^{(j)}(U) \leq (1 - \epsilon^{-10})\Phi^{(j)}(U)$ by Corollary 18. Since the considered color i is not a *Sink*, we have $\Phi^{(j)}\tilde{\chi}^{-1}(i) \leq \|\Phi^{(j)}\chi^{-1}\|_\infty$ throughout the algorithm. So $\Phi^{(j)}\chi_1^{-1}(i)$ is as required for $i \notin \mathit{Sink}$.

Claim 3. For non-sink colors i and all $j \in [r]$, it holds $\Phi^{(j)}\chi_0^{-1}(i) \leq \|\Phi^{(j)}\chi^{-1}\|_\infty$ and $\Phi^{(j)}\chi_1^{-1}(i) \leq (1 - \epsilon^{10})\|\Phi^{(j)}\chi^{-1}\|_\infty$

To show the corresponding claim for sink colors, we need to estimate the $\Phi^{(j)}$-weight of the parts $X \subseteq W$ that get transfered from *Source* colors to *Sink* colors.

Two types of parts are considered by the algorithm. By Corollary 16, we have for any part X that gets inserted into *Buffer*,

$$\Phi^{(j)}(X) \leq 18\epsilon/M \cdot \|\Phi^{(j)}\chi^{-1}\|_\infty, \tag{13}$$

and by Corollary 17, any parts X that is painted in step (2.) of procedure AddTo satisfies

$$\Phi^{(j)}(X) \leq 18\epsilon \cdot \|\Phi^{(j)}\chi^{-1}\|_\infty. \tag{14}$$

Since the parts of the second type are much more expensive (in terms of $\Phi^{(j)}$-weight) than the parts of the first type only the following observations allows us to derive the required bound on $\Phi^{(j)}\chi_1^{-1}(i)$ for $i \in \mathit{Sink}$.

Key-Observation: For a sink color i, either a) all received parts are of the first type, or b) throughout the algorithm color i received only one part and hence $\Psi\tilde{\chi}^{-1}(i) \leq \epsilon\Psi^* + 3\epsilon\Psi^*$, because this parts had Ψ-weight at most $3\epsilon\Psi^*$.

We show the required bound on $\Phi^{(j)}\chi_1^{-1}(i)$ by distinguishing these two cases.

Case a): It is an invariant of the algorithm that sink colors have color classes of Ψ-weight at most $\Psi^* + 3\epsilon\Psi^* \leq 2\Psi^*$. And since each received part has Ψ-weight at least $\epsilon\Psi$, a sink color i can receive no more than $2/\epsilon$ parts. In the current case, all of these parts have $\Phi^{(j)}$-weight at most $18\epsilon/M \cdot \|\Phi^{(j)}\chi^{-1}\|_\infty$. Hence in step (5.) of procedure Shrink, it holds $\Phi^{(j)}(U) \leq (1+36/M)\|\Phi^{(j)}\chi^{-1}\|_\infty$ for the i-th color class $U = \tilde{\chi}^{-1}(i)$. Since $\Psi^*/\Psi(U) \geq 1/2$, we have by Corollary 18

$$\Phi^{(j)}(U \backslash X_i)/\|\Phi^{(j)}\chi^{-1}\|_\infty \leq (1 - \epsilon/18)(1 + 36\epsilon^5) \leq 1 - \epsilon^{10}$$

for sufficiently small $\epsilon > 0$. Thus the $\Phi^{(j)}$-weight of $\chi_1^{-1}(i) = U \setminus X_i$ is as required by the lemma.

Case b): By our key observation, we have $\Psi^*/\Psi(U) \geq 1/4\epsilon$ for the i-th color class $U = \tilde{\chi}^{-1}(i)$ at the end of procedure Shrink. It also holds $\Phi^{(j)}(U) \leq (1 + 18\epsilon)\|\Phi^{(j)}\chi^{-1}\|_\infty$, since only one part was received by color i in the course of the algorithm. By Corollary 18, the $\Phi^{(j)}$-weight of $\chi_1^{-1}(i) = U \setminus X_i$ satisfies for sufficiently small $\epsilon > 0$,

$$\Phi^{(j)}(U \backslash X_i)/\|\Phi^{(j)}\chi^{-1}\|_\infty \leq (1 - \tfrac{\epsilon}{9} \cdot \tfrac{1}{4\epsilon})(1 + 18\epsilon) \leq 1 - \epsilon^{10}$$

This case distinction showed for every $i \in Sink$:

Claim 4. $\Phi^{(j)}\chi_1^{-1}(i) \leq (1 - \epsilon^{10})\|\Phi^{(j)}\chi^{-1}\|_\infty$.

The discussion for case a) above, implies that any sink color i receives at most constantly many parts. Then it follows from relations (13) and (14), that $\Phi^{(j)}\chi_1^{-1}(i) = O_M(\|\Phi^{(j)}\chi^{-1}\|_\infty)$.

The fact that any color class is altered only constantly often also shows that the algorithm can be implemented to run in time $O_M(t(|G[W]|))$, provided that appropriate data structures are used. For example in step (2.) of procedure Shrink, the colors with $\Psi\tilde{\chi}^{-1}(i) > M/2 \cdot \Psi^*$ should be maintained by a stack, so that such colors can be found in constant time. □

6 Splittability of grid graphs

For the case of unit-costs, many graph classes like fixed-minor free graphs and finite element meshes have bounded p-splittability for some $p > 1$ (cf. Appendix A.3, Remark 36). In case of arbitrary edge costs, only planar graphs were known to have splitting sets of low costs.

There is a naive way to generalize existing splittability results for the unit costs case to the case of arbitrary edge costs. Obviously, it holds $\sigma_p(G, c) \leq \sigma_p(G, \mathbb{1}_E) \cdot \|c\|_\infty \cdot \|1/c\|_\infty$ for every graph $G = (V, E)$ and arbitrary edge costs $c \colon E \to \mathbb{R}_{>0}$ (assuming without loss of generality $c(e) > 0$ for all edges $e \in E$). In this section, we show that the situation for d-dimensional "grid graphs" is better. A *grid graph* in a d-dimensional space is a graph $G = (V, E)$ with $V \subseteq \mathbb{Z}^d$ and $\|\mathbf{x} - \mathbf{y}\|_1 = 1$ for all edges $\{\mathbf{x}, \mathbf{y}\} \in E$. The theorem of this section implies that $\sigma_p(G, c) = O_d(\log^{1/d}(\phi) \cdot \sigma_p(G, \mathbb{1}))$ if G is a d-dimensional grid, $p = d/(d-1)$ and $\phi := \|c\|_\infty \cdot \|1/c\|_\infty$ is the ratio of the maximum cost of an edge to the minimum cost of an edge.

Although grids form a very restrictive graph family, many graphs arising in practical applications, e.g., in climate simulation (cf. Introduction), are "close" to grid graphs and might be embedded into grids such that boundary costs are preserved up to constant factors.

Furthermore, the results in this section can be seen as a starting point for further investigations of the splittability for more general non-planar graphs with arbitrary edge costs. Note that for $d \geq 3$, the class of d-dimensional grids does not exclude any minor and hence is "far" from being planar.

We state the main theorem of this section.

Theorem 19. *Let $G = (V, E)$ be a d-dimensional grid graph with edge costs $c : E \to \mathbb{R}_{>0}$ and vertex weights $w : V \to \mathbb{R}_+$. Then, for all splitting values w^* with $0 \leq w^* \leq \|w\|_1$, there exists a w^*-splitting set $U \subseteq V$ of cost*

$$O(d \cdot \log^{1/d}(\phi + 1) \cdot \|c\|_p)$$

where $p := d/(d-1)$ and $\phi := \max_E c / \min_E c$ is the fluctuation *of c. Such a splitting set can be computed in time $O(n \log \phi)$.*

And since the class of d-dimensional grid graphs is closed under taking subgraphs, we have

$$\sigma_p(G, c) = O_d(\log^{1/d}(\phi + 1)).$$

In the following, let $G = (V, E)$, $c\colon E \to \mathbb{R}_{>0}$, $w\colon V \to \mathbb{R}_+$, and w^* be as in Theorem 19. We aim for a recursive algorithm to find a w^*-splitting set in G with small boundary cost.

The idea of the algorithm is as follows. We consider a *coarser* graph that is obtained by identifying vertices of G, i.e., for a mapping $\varphi\colon V \to \mathbb{Z}^d$, the coarser graph $G/\varphi = (V/\varphi, E/\varphi)$ has the non-empty pre-images $\varphi^{-1}(\mathbf{a}) = \{\mathbf{x} \in V \mid \varphi(\mathbf{x}) = \mathbf{a}\} \in V/\varphi$ as nodes and contains an arc $\{\varphi^{-1}(\mathbf{a}), \varphi^{-1}(\mathbf{b})\} \in E/\varphi$ if and only if an edge $\{\mathbf{x}, \mathbf{y}\}$ of G connects the two disjoint sets $\varphi^{-1}(\mathbf{a})$ and $\varphi^{-1}(\mathbf{b})$. Formally,

$$V/\varphi := \Big\{\varphi^{-1}(\mathbf{a}) \neq \emptyset \mid \mathbf{a} \in \mathbb{Z}^d\Big\},$$

$$E/\varphi := \Big\{\{Q, R\} \subseteq V/\varphi \mid \mathbf{x} \in Q, \mathbf{y} \in R, \{\mathbf{x}, \mathbf{y}\} \in E, Q \neq R\Big\}$$

The weights and costs are translated to G/φ in a straight-forward manner; we define $w/\varphi(Q) := w(Q)$ for each $Q \in V/\varphi$ and $c/\varphi(Q, R) := \sum_{\mathbf{x} \in Q, \mathbf{y} \in R} c(\mathbf{x}, \mathbf{y})$. In this coarser graph, which shall also be a grid, we then use a trivial algorithm to find a splitting set $\mathcal{S} \subseteq V/\varphi$ and a node $Q \in V/\varphi \setminus \mathcal{S}$ with $w/\phi(\mathcal{S}) \leq w^* < w/\phi(\mathcal{S}) + w(Q)$. So $\mathcal{S}$ has the desired weight w^* up to the weight of Q. Now the idea is to proceed recursively in order to compute a $(w^* - w/\varphi(\mathcal{S}))$-splitting set $U' \subseteq Q$ in $G' := G[Q]$. Then, $U := \bigcup \mathcal{S} \cup U'$ will be the required w^*-splitting set in G.

To bound the boundary cost ∂U of the resulting splitting set, our reasoning will be as follows. Since any boundary edge $\delta(U)$ is contained in either $\delta(\bigcup \mathcal{S})$, $\delta(Q)$, or $\delta_{G[Q]}(U')$, the boundary cost satisfies $\partial U \leq \partial(\bigcup \mathcal{S} \cup Q) + \partial_Q U'$. In our analysis we shall use the crude over-estimate $\partial(\bigcup \mathcal{S} \cup Q) \leq \|c/\varphi\|_1$ to deduce that ∂U is at most $\|c/\varphi\|_1 + \partial_Q U'$.

As we will see, there are canonical choices of φ that guarantee a good upper bound on $\|c/\varphi\|_1$ (cf. Lemma 20). However, to make the recursion work, we need to ensure that a splitting set in $G' = G[Q]$ is somehow easier to obtain and less costly. It seems difficult to do so, and therefore we circumvent this issue. The recursive instance does not use the same edge costs c but "reduced" edge costs $c' \colon E' \to \mathbb{R}_{>0}$ with $c'_e := (c_e - 1)/2$ for all edges with $c_e > 1$. All edges with $c_e \leq 1$ are discarded in G'. With this choice of c', we end up with an empty graph after $O(\log \|c\|_\infty)$ levels of recursion. So there is some notable progress when going to recursive instances. On the other hand, we need to take into account the edges of G that were discarded in G', when we want to deduce a bound on $\partial_Q U'$ from $\partial'(U')$, where ∂' denotes the boundary costs in G' with respect to c'. We use again a rough estimate $\partial_Q U' \leq |\delta_{G[Q]}(U')| + 2\partial' U'$, which holds with equality only if $\delta_{G[Q]}(U')$ contains no edge e with $c(e) < 1$. In order to control $|\delta_{G[Q]}(U')|$, we shall use specific properties of $Q \in V/\varphi$ that follow from our choice of G/φ, and we will exploit an invariant of our recursive algorithm, namely that the computed splitting sets are "monotone" (cf. Lemma 21-23).

Obtaining a coarser graph with low edge costs. Grid graphs can be coarsened nicely in a geometrically intuitive way. We partition the d-dimensional space $\mathbb{R}^d$ in "half-open" hyper-cubes $\mathbf{x} + [0,\, \ell)^d$ of measure ℓ^d, where $\mathbf{x} + Y := \{\mathbf{x} + \mathbf{y} \mid \mathbf{y} \in Y\}$ denotes the Minkowski sum of $\mathbf{x} \in \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^d$. This partition is in such a way that each face of a cube in the partition is also the face of another cube. Then, we identify all vertices of grid G that lie in the same cube.

Formally, we define mappings

$$\varphi^{(\ell)}_\alpha \colon \mathbb{Z}^d \to \mathbb{Z}^d, \quad \mathbf{a} \mapsto \lfloor (\mathbf{a} + (\alpha - 1) \cdot \mathbb{1}_d)/\ell \rfloor$$

for positive integers $\alpha, \ell \in \mathbb{N}$, where $\mathbb{1}_d \in \mathbb{R}^d$ is the vector of all ones and $\lfloor \mathbf{x} \rfloor$ is the integer vector with components $\lfloor x_i \rfloor$. Then, the graph $G/\varphi^{(\ell)}_\alpha$ is a coarser graph of the kind described above.

We say that a graph G/φ is *ℓ-coarse* if $\varphi = \varphi^{(\ell)}_\alpha$ for some $\alpha \in [\ell]$. The lemma below observes that we can always find an ℓ-coarse graph with low edge costs.

Lemma 20. *For each positive integer ℓ, there exists an ℓ-coarse graph G/φ with*

$$\|c/\varphi\|_1 \leq \|c\|_1/\ell$$

Proof. Since each edge of G accounts for exactly one of the cost functions $c/\varphi^{(\ell)}_1$ through $c/\varphi^{(\ell)}_\ell$, we have $\|c/\varphi^{(\ell)}_1\|_1 + \ldots + \|c/\varphi^{(\ell)}_\ell\|_1 = \|c\|_1$. Hence, $\|c/\varphi^{(\ell)}_\alpha\|_1 \leq \|c\|_1/\ell$ for some $\alpha \in [\ell]$. □

We say that an ℓ-coarse graph G/φ is *cheap* if $\|c/\varphi\|_1 \leq \|c\|_1/\ell$.

Monotone sets. For a set $Q \subseteq \mathbb{Z}^d$, we say that a subset $W \subseteq Q$ is *monotone* (in Q), if for all $\mathbf{x} \in Q$ and $\mathbf{y} \in W$ with $\mathbf{x} \leq \mathbf{y}$, it holds $\mathbf{x} \in Q$, where $\mathbf{x} \leq \mathbf{y}$ means that we have $x_i \leq y_i$ for each component of $\mathbf{x}$ and $\mathbf{y}$.

We shall see later that it is an invariant of our recursive algorithm that the computed splitting set U' of $G' = G[Q]$ is monotone in $Q \in V/\varphi$. The next lemma allows us to bound $|\delta_{G[Q]}(U')|$, provided that G/φ is ℓ-coarse.

Lemma 21. *Let $Q \subseteq \mathbb{Z}^d$ be a node of an ℓ-coarse graph G/φ, i.e., $Q \subseteq \mathbf{x} + [0, \ell)^d$ for some $\mathbf{x} \in \mathbb{Z}^d$.*

Then for any monotone set W in Q, it holds $|\delta_{G[Q]}(W)| \leq d\ell^{d-1}$.

Proof. For each edge $\mathbf{ab} \in \delta_{G[Q]}(W)$, consider the line $L = \{\lambda\mathbf{a} + (1-\lambda)\mathbf{b} \mid \lambda \in \mathbb{R}\}$ through this edge. By monotonicity of W, no other edge of $\delta_{G[Q]}(W)$ is contained in L. On the other hand, we can uniquely identify L by the point in $Q \cap L$ with the smallest coordinates (the point at which L "leaves" Q). Notice that the points in $L \cap Q$ can differ only in one component and therefore agree in $d-1$ components. So if the direction of L is fixed, the "lowest" point in $L \cap Q$ is uniquely determined by its coordinates in $d-1$ components.

Any of these "leaving" points can be generated as follows. First, we select one of the d components, say the i-th component, that needs to be smallest. Then, there are at most ℓ^{d-1} choices for the entries of the other components. No further choices remain, since the i-th component needs to be smallest. Thus, there are at most $d \cdot \ell^{d-1}$ edges in the cut $\delta_{G[Q]}(W)$. □

The lemmata below shall imply the invariant that U' is a monotone set in Q.

Lemma 22. *If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is a lexicographic ordering of a set $S \subseteq \mathbb{Z}^d$, then for all $i \in [n]$, the subset $\{\mathbf{a}_1, \ldots, \mathbf{a}_i\}$ is monotone in S.*

Proof. A vector $\mathbf{x}$ is lexicographically less than a vector $\mathbf{y}$ if it holds $\mathbf{x} \leq \mathbf{y}$. □

For every node $R = \varphi^{-1}(\mathbf{a}) \subseteq \mathbb{Z}^d$ of G/φ, we conveniently define $\varphi(R) := \mathbf{a}$.

Lemma 23. *Let G/φ be an ℓ-coarse graph of grid $G = (V, E)$ and $Q_1, \ldots, Q_i \in V/\varphi$ be a sequence of distinct nodes of G/φ. Suppose both $\{\varphi(Q_1), \ldots, \varphi(Q_{i-1})\}$ and $\{\varphi(Q_1), \ldots, \varphi(Q_i)\}$ are monotone sets in $\varphi(V/\varphi) := \{\varphi(R) \mid R \in V/\varphi\}$. Then for any monotone set W in Q_i, the set $Q_1 \cup \ldots \cup Q_{i-1} \cup W$ is monotone in V.*

Proof. Notice that $Q_1 \cup \ldots \cup Q_{i-1}$ is monotone in V, since $\{\varphi(Q_1), \ldots, \varphi(Q_{i-1})\}$ is monotone in $\varphi(V/\varphi)$ and φ is monotone, i.e., $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$ for all $\mathbf{x} \leq \mathbf{y}$. And since W is monotone in Q_i, it only remains to verify that for all $\mathbf{y} \in W$ and $\mathbf{x} \in V \setminus W$ with $\mathbf{x} \leq \mathbf{y}$, it holds $\mathbf{x} \in Q_j$ for some $j < i$. But this follows from the monotonicity of $\{\varphi(Q_1), \ldots, \varphi(Q_i)\}$ in $\varphi(V/\varphi)$ and from the monotonicity of φ. □

Final algorithm. We now have gathered all ingredients of the algorithm for computing monotone splitting sets in grid graphs. However, we have not yet determined how ℓ should be chosen. Remember that we derived the relation

$$\partial U \leq \|c/\varphi\|_1 + |\delta_{G[Q]}(U')| + 2\partial' U'$$

between the cost of U in G and the cost of U' in G'. From Lemma 20 and Lemma 21, it would follow $\partial U \leq \|c\|_1/\ell + d\ell^{d-1} + 2\partial' U'$. When we choose $\ell := (\|c\|_1/d)^{1/d}$, the expression is minimized and we obtain

$$\partial U \leq 2d^{1/d} \|c\|_1^{1-1/d} + 2 \cdot \partial' U'. \tag{15}$$

Procedure GridSplit (grid graph $G = (V, E)$, edge costs $c \colon E \to \mathbb{R}_{>0}$, $w^* \in \mathbb{R}_+$)

(1.) Compute a cheap ℓ-coarse graph G/φ of G with $\ell := \max\{\lceil (\|c\|_1/d)^{1/d} \rceil, 1\}$ (Lem. 20)

(2.) Find an ordering $Q_1, \ldots, Q_q$ of the vertices of G/φ such that $\varphi(Q_j)$ is lexicographically less than $\varphi(Q_{j+1})$ for all $j \in [q-1]$

(3.) Determine a set $\mathcal{S} = \{Q_1, \ldots, Q_{i-1}\}$ with $w(\bigcup \mathcal{S}) \le w^* < w(\bigcup \mathcal{S}) + w(Q_i)$.
(4.) *Trivial case:* If $\ell = 1$ then return a w^*-splitting set among $\bigcup \mathcal{S}$ and $\bigcup \mathcal{S} \cup Q_i$
(5.) Recursively compute a monotone splitting set

$$U' := \textsc{GridSplit}\Big(G', c', w^* - w(\bigcup \mathcal{S}\Big),$$

where $G' := G[Q] \setminus \{e \in E \mid c(e) \le 1\}$ and $c'_e := (c_e - 1)/2$ for all $e \in E(G')$
(6.) Return the w^*-splitting set $U := \bigcup \mathcal{S} \cup U'$

Notice that GridSplit terminates after $O(\log \|c\|_\infty)$ levels of recursion. The maximum cost of an edge decreases by a factor of at least 2 with every level. And as soon as $\|c\|_\infty \le 1$, we have $\|c'\|_\infty = 0$ and terminate in the next level as $\ell' = \max\{\lceil (\|c'\|_1/d)^{1/d}\rceil, 1\} = 1$.

Before bounding the cost of the splitting set computed by GridSplit , we show that the splitting set is indeed monotone in V. The monotonicity of the splitting set shall follow from Lemma 22 and Lemma 23.

Lemma 24. *The splitting set computed by* GridSplit *is monotone in V.*

Proof. By induction on $\|c\|_\infty$.

In the case $\ell = 1$, the 1-coarse graph G/φ coincides with G, since φ is the identity. Now $\varphi(Q_1), \ldots, \varphi(Q_q)$ is a lexicographic ordering of V. By Lemma 22, both $\bigcup \mathcal{S} = \{\varphi(Q_1), \ldots, \varphi(Q_{i-1})\}$ and $\bigcup \mathcal{S} \cup Q_i = \{\varphi(Q_1), \ldots, \varphi(Q_i)\}$ are monotone in $\varphi(V/\varphi) = V$.

For $\ell > 1$, the splitting set U' is monotone in Q by induction hypothesis. Also, it holds that both $\{\varphi(Q_1), \ldots, \varphi(Q_{i-1})\}$ and $\{\varphi(Q_1), \ldots, \varphi(Q_i)\}$ are monotone in $\varphi(V/\varphi)$ by Lemma 22. So we can apply Lemma 23 and obtain that $\bigcup \mathcal{S} \cup U' = Q_1 \cup \ldots \cup Q_{i-1} \cup U'$ is monotone in V. ☐

Now we are armed to show our first (technical) bound on the boundary cost ∂U, which is obtained by unfolding the recurrence (15) of procedure GridSplit.

Lemma 25. *If* GridSplit *is applied to a grid graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{R}_+$, then the returned monotone splitting set $U \subseteq V$ satisfies*

$$\partial U \le 2^d d^{1/d}\Big(\|c\|_\infty + 1 + \sum_{0 \le i \le \log(\|c\|_\infty + 1)} 2^{i/d} \|c_{|E_i}\|_1^{1-1/d}\Big)$$

where $E_i := \{e \in E \mid c(e) \ge 2^i - 1\}$.

Proof. By induction on $\lfloor \log(\|c\|_\infty + 1)\rfloor$.

For $\ell = 1$, we have $\|c\|_1 \le d^{1/d}$ and therefore $\partial U \le d^{1/d}$. So, we can assume $\ell = \lceil (\|c\|_1/d)^{1/d}\rceil > 1$ and thus $\|c\|_1/d \le \ell^d \le 2^d \|c\|_1/d$.

In case of $\|c\|_\infty \le 1$, it holds $\partial_Q U' \le |\delta_{G[Q]}(U')|$. Since U' is monotone by Lemma 24 in Q, we have by Lemma 21 that $|\delta_{G[Q]}(U')| \le d\ell^{d-1} \le 2^{d-1} d^{1/d} \|c\|_1^{1-1/d}$. As G/φ is a cheap ℓ-coarse grid, the edge costs $\|c/\varphi\|_1$ are at most $\|c\|_1/\ell \le d^{1/d}\|c\|_1^{1-1/d}$ and thus we have $\partial U \le \|c/\varphi\|_1 + \partial_Q U' \le 2^d d^{1/d} \|c\|_1^{1-1/d}$ as required.

For the rest of the proof, we can assume $\|c\|_\infty > 1$ and $\partial U \le 2^d d^{1/d} \|c\|_1^{1-1/d} + 2\partial' U'$ where ∂' are the boundary costs in G' with respect to edge costs c'. By induction hypothesis, it holds $\partial' U' \le 2^d d^{1/d}(\|c'\|_\infty + 1 + \sum_{0 \le i \le r} 2^{i/d} \|c'_{|E'_i}\|_1^{1-1/d})$, where $r := \log(\|c\|'_\infty + 1) =$

$\log(\|c\|_\infty + 1) - 1$ and $E'_i := \{e \in E' \mid 2^i - 1 \le c'_e = (c_e - 1)/2\} = E_{i+1}$. So, it holds $\|c'_{|E'_i}\|_1 \le \|c_{|E_{i+1}}\|_1/2$ and thus $2 \cdot 2^{i/d}\|c'_{|E'_i}\|_1^{1-1/d} \le 2^{(i+1)/d}\|c_{|E_{i+1}}\|_1^{1-1/d}$. Therefore, we have

$$\partial U \le 2^d d^{1/d}\|c\|_1^{1-1/d} + 2^d d^{1/d}\Big(\|c\|_\infty + 1 + \sum_{0\le i\le r} 2^{i+1/d}\|c_{|E_{i+1}}\|_1^{1-1/d}\Big)$$

as required. ☐

Using Hölder's inequality, the lemma below arrives at a bound on $\sum_i 2^{i/d}\|c_{|E_i}\|_1^{1-1/d}$. Notice that by scaling the edge cost, we can achieve $\|1/c\|_\infty = 1$ and therefore $\phi = \|c\|_\infty \cdot \|1/c\|_\infty = \|c\|_\infty$. So the next lemma implies the bounds on the p-splittability of grid graphs from Theorem 19.

Lemma 26. *For edge costs $c\colon E \to \mathbb{R}_{>0}$ with $\|1/c\|_\infty = 1$, it holds*

$$\sum_{i=0}^{\lfloor \log(\|c\|_\infty+1)\rfloor} 2^{i/d}\|c_{|E_i}\|_1^{1-1/d} = O_d\Big((\log^{1/d}(2\|c\|_\infty) \cdot \|c\|_{d/(d-1)}\Big)$$

Proof. Let $r\colon E \to \mathbb{N}_0$ be the function that assigns each edge $e \in E$ to the largest index $r(e) := \lfloor \log(c_e + 1)\rfloor$ such that $e \in E_{r(e)}$. Let $s := \|r\|_\infty \le \log(2\|c\|_\infty)$ denote the largest index with $E_s \ne \emptyset$. Since $c_e \ge 1$ for each edge $e \in E$, it holds $r(e) \le \log(2c_e)$ and thus

$$c_e \cdot \sum_{0\le i\le r(e)} 2^{i/(d-1)} = O_d(c_e \cdot 2^{r(e)/(d-1)}) = O_d(c_e^{d/(d-1)}) \tag{16}$$

Hence, the following sum satisfies

$$\sum_{0\le i\le s} 2^{i/(d-1)}\|c_{|E_i}\|_1 = \sum_{e\in E} c_e \sum_{0\le i\le r(e)} 2^{i/(d-1)} \overset{(16)}{=} O_d(\sum_{e\in E} c_e^{d/(d-1)}) \tag{17}$$

Using Hölder's inequality and relation (17) yields for $p = d/(d-1)$ (and $q = d$),

$$\sum_{i=0}^{s} 1 \cdot (2^{i/(d-1)}\|c_{|E_i}\|_1)^{1-1/d} = \|r\|_\infty^{1/d} \cdot (\sum_{i=0}^{s} 2^{i/(d-1)}\|c_{|E_i}\|_1)^{1-1/d} \overset{(17)}{=} O_d(s^{1/d}\|c\|_p) \tag{18}$$

From relation (18), the lemma follows as $2^{i/d}\|c_{|E_i}\|_1^{1-1/d} = 1 \cdot (2^{i/(d-1)}\|c_{|E_i}\|_1)^{1-1/d}$. ☐

We conclude the section with an analysis of the running time of procedure GridSplit.

Lemma 27. *Procedure* GridSplit *runs in time $O(m \log \phi)$ for a connected grid graph G of size m with edge costs c of fluctuation ϕ.*

Proof. We assume that the edge costs are scaled in such a way that the minimum edge cost is equal to 1 and so it holds $\phi = \|c\|_\infty$. Then the number of iterations is bounded by $O(\log \phi)$.

It remains to show that the running time of one iteration is linear in the size of the grid. Steps (3)-(6) are easily seen to run in linear time. In step (1), finding a cheap ℓ-coarse graph is trivial if ℓ is much larger than the size of G, because for a connected grid, one of the mappings $\varphi_\alpha^{(m)}$ assigns the same point to all grid vertices.

So we can assume $\ell = O(m)$ for finding a cheap ℓ-coarse graph in step (1). For each edge $\mathbf{ab} \subseteq \mathbb{Z}^d$ with $\|\mathbf{a}-\mathbf{b}\|_1 = 1$, we can determine in constant time the index $\alpha \in [\ell]$ with $\varphi_\alpha^{(\ell)}(\mathbf{a}) \neq \varphi_\alpha^{(\ell)}(\mathbf{b})$. Thus, the function $f\colon [\ell] \to \mathbb{R}_+$ with $f(\alpha) := \|c/\varphi_\alpha^{(\ell)}\|_1$ can be computed in linear time by scanning through all edges of G. Now we can find a cheap ℓ-coarse graph by finding the minimum of $f(\alpha)$ over all $\alpha \in [\ell]$.

In step (2) of GridSplit , we need to sort points from $\mathbb{Z}^d$ by lexicographic order. The range of the coordinates of the considered points is polynomial, since G is connected (in fact, the range is linear). So we can use *radix sort* to find a lexicographic ordering in linear time. □

7 Conclusion

We showed that any graph with edge costs and vertex weights can be partitioned into a given number of almost equally-weighted parts in such a way that the maximum boundary cost is small, provided that the graph has small splittability.

Using an observation from [4], namely that the boundary cost function can approximately be modeled as a (dynamic) weight-function on the vertices, we reduced the *min-max boundary decomposition* problem to a *multi-balanced* partitioning problem. For the case of arbitrary edge costs, it was necessary to balance the partition also with respect to the *splitting cost measure*.

Finally, we developed an algorithm based on a "shrink-and-conquer"-approach for improving the weight-balancedness of a partition while maintaining the balanced with respect to a number of other measures, including the boundary cost function.

We remark that, using our general framework, one can devise a multi-balanced version of Theorem 4: Every graph G with edge costs c, measures Ψ and $\Phi^{(1)}$ through $\Phi^{(r)}$ can be partitioned into k parts such that 1.) the Ψ-weight of each part differs from the average by at most $(1-1/k)\|\Psi\|_\infty$, 2.) for each measure $\Phi^{(j)}$, the maximum $\Phi^{(j)}$-weight of the partition is at most proportional to the average $\Phi^{(j)}$-weight, and 3.) the maximum boundary cost of the partition is at most proportional to $\sigma_p \cdot (\|c\|_p/k^{1/p} + \Delta_c)$.

A possible direction of further work suggested by this thesis is to investigate the question whether more general graphs have separator theorems when one allows arbitrary edge costs. Only planar graphs and grid graphs are known to have separator theorems in this case. And our separator theorem for grid graphs still leaves space for improvement.

Acknowledgment

I would like to thank my advisor Peter Sanders for suggesting the problem, and for many valuable discussions about this work.

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A Further proofs

A.1 Shrinking cuts

In the following we shall show the lemmata needed for deriving Corollaries 16-18.

Let the function $\pi\colon V \to \mathbb{R}_+$ be the p-splitting cost measure (cf. Definition 10) of the graph $G = (V, E)$ with measures Ψ, and $\Phi^{(1)}$ through $\Phi^{(r)}$.

Lemma 28. *For every $U \subseteq V$ and $\gamma \in [0, 1]$ with $\|\Psi\|_\infty/\Psi(U) \leq \gamma/3r$, there exists a partition $\{X_1, \ldots, X_\ell\}$ of U with $r/\gamma \leq \ell \leq 3r/\gamma$ and $\partial_U X_i = O(r/\gamma \cdot \pi^{1/p}(U))$ such that*

$$\frac{\gamma}{3r} \leq \frac{\Psi(X_i)}{\Psi(U)} \leq \frac{\gamma}{r} \quad \text{for } i \leq \ell.$$

Proof. We just need to apply the following procedure with $\psi^* := \gamma/3r \cdot \Psi(U)$.

Procedure ITERATIVEPARTITION (vertex set $U \subseteq V$, $\psi^* \in \mathbb{R}_+$)

// *Precondition:* $\|\Psi\|_\infty \leq \psi^*$

(1.) Start with $X \leftarrow U$ and $i \leftarrow 1$

(2.) Until $\Psi(X) \leq 3\psi^*$ repeat: // *Invariant:* $\Psi(X) \geq \psi^*$

   (a) Let X_i be a splitting set in $G[X]$ with $\partial_X X_i \leq \pi^{1/p}(X)$ and $\psi^* \leq \Psi(X_i) \leq \psi^* + \|\Psi\|_\infty$

   (b) Update $X \leftarrow X \setminus X_i$ and increment $i \leftarrow i + 1$

(3.) Set $\ell := i$ and $X_\ell := X$

(4.) Return the partition $\{X_1, \ldots, X_\ell\}$ of U

All parts X_i, especially X_ℓ, fulfill the condition $\psi^* \le \Psi(X_i) \le 3\psi^*$. Hence, it holds $\ell \le \Psi(U)/\psi^* = 3r/\gamma$ and $\ell \ge \Psi(U)/3\psi^* = r/\gamma$. The total cost of edges that are cut by the algorithm does not exceed $\ell \cdot \pi^{1/p}(U)$. It follows $\partial_U X_i = O(r/\gamma \cdot \pi^{1/p}(U))$. ☐

Lemma 29. *For U and γ as in Lemma 28, there exists a subset X of U with $\partial_U X = O(r/\gamma \cdot \pi^{1/p}(U))$ such that*

$$\frac{\gamma}{3r} \le \frac{\Psi(X)}{\Psi(U)} \le \frac{\gamma}{r}, \quad \frac{\Phi^{(j)}(X)}{\Phi^{(j)}(U)} \le \gamma \text{ for all } j \in [r].$$

Proof. By Lemma 28 there exists $\ell \ge r/\gamma$ disjoint subsets X_i of U with Ψ-weight as required. By the pigeonhole-principle, one of those parts X_i has to fulfill $\Phi^{(j)}(X_i) \le \gamma \cdot \Phi^{(j)}(U)$ for all $j \in [r]$. ☐

The lemma below is dual to Lemma 29.

Lemma 30. *For U and γ as in Lemma 28, there exists a subset X of U with $\partial_U X = O(r^2/\gamma \cdot \pi^{1/p}(U))$ such that*

$$\gamma \le \frac{\Psi(X)}{\Psi(U)} \le \gamma + \frac{\|\Psi\|_\infty}{\Psi(U)}, \quad \frac{\Phi^{(j)}(X)}{\Phi^{(j)}(U)} \ge \frac{\gamma}{3r} \text{ for all } j \in [r].$$

Proof. By Lemma 28 there exists a partition of U into parts $X_1, \dots, X_\ell$ with $\ell \le 3r/\gamma$ and $\Psi(X_i) \le \gamma/r \cdot \Psi(U)$. Without loss of generality we may assume that for all $j \in [r]$, there is a part X_i with index $i \le r$ and maximum $\Phi^{(j)}$-weight among all parts. Formally,

$$\max_{1 \le i \le \min\{r,\ell\}} \Phi^{(j)}(X_i) = \max_{1 \le i \le \ell} \Phi^{(j)}(X_i) \ge \Phi^{(j)}(U)/\ell \text{ for all } j \in [r].$$

It follows for the union $\bar{X} := X_1 \cup \dots X_r$ of the first r parts that $\Phi^{(j)}(\bar{X})/\Phi^{(j)}(U) \ge \gamma/3r$ for all $j \in [r]$. The Ψ-weight of $\bar{X}$ cannot exceed $r \cdot \gamma/r \cdot \Psi(U) = \gamma\Psi(U)$. Also the cost of the boundary edge of $\bar{X}$ within $G[U]$ satisfies $\partial_U(X) = O(r \cdot r/\gamma \cdot \pi^{1/p}(U))$.

To fulfill the constraint $\Psi(X) \ge \gamma \cdot \Psi(U)$ we need to find a subset S of $U \backslash \bar{X}$ such that $\Psi(\bar{X} \cup S) \ge \gamma \cdot \Psi(U)$. So let $S \subseteq U \backslash \bar{X}$ be a splitting set in $G[U \backslash \bar{X}]$ with $\partial_{U \backslash \bar{X}}(S) \le \pi^{1/p}(U)$ and $\gamma \cdot \Psi(U) \le \Psi(S) + \Psi(\bar{X}) \le \gamma \cdot \Psi(U) + \|\Psi\|_\infty$. Then $X := \bar{X} \cup S$ is a subset of U that fulfills all requirements of the lemma. ☐

From Lemma 29 and Lemma 30, we draw the three corollaries below. Let $\epsilon > 0$ be sufficiently small and $M := 1/\epsilon^5$, where the precise meaning of "sufficiently" depends only on r. So in Section 5 we can assume that ϵ and M are absolute constants. Moreover, let Ψ^* be a real number between 0 and $\|\Psi\|_{avg}$ such that $\|\Psi\|_\infty \le \epsilon^5\Psi^*$. This condition corresponds to the condition on $\|\Psi\|_\infty$ in the definition of shrinking procedures (cf. Definition 13).

In order to achieve a geometric decrease of the boundary costs, we choose the measure $\Phi^{(r)}\colon V \to \mathbb{R}_+$ such that $\Phi^{(r)}(v) := c(\delta(v) \cap \delta(U))$, where $U \subseteq V$ is as in the corollaries below. Then corollaries 16-18 are instantiations of the corollaries below for $r = 3$.

Corollary 31. *For every $U \subseteq V$ with $M/2 \le \Psi(U)/\Psi^* \le M$, there exists a subset X of U with $\partial_U X = O_M(\pi^{1/p}(U))$ and $\epsilon \le \Psi(X)/\Psi^* \le 3\epsilon$ such that*

$$\begin{aligned} \Phi^{(j)}(X) &\le (6r\epsilon/M) \cdot \Phi^{(j)}(U), \\ \partial X &\le (6r\epsilon/M) \cdot \partial U + O_M(\pi^{1/p}(U)). \end{aligned}$$

Corollary 32. *For every* $U \subseteq V$ *with* $1/2 \leq \Psi(U)/\Psi^* \leq M$*, there exists a subset* X *of* U *with* $\partial_U X = O_M(\pi^{1/p}(U))$ *and* $\epsilon \leq \Psi(X)/\Psi^* \leq 3\epsilon$ *such that*

$$\begin{aligned}\Phi^{(j)}(X) &\leq 6r\epsilon \cdot \Phi^{(j)}(U),\\ \partial X &\leq 6r\epsilon \cdot \partial U + O_M(\pi^{1/p}(U)).\end{aligned}$$

Corollary 33. *For every* $U \subseteq V$ *with* $\epsilon \leq \Psi(U)/\Psi^* \leq M$*, there exists a subset* X *of* U *with* $\partial_U X = O_M(\pi^{1/p}(U))$ *and* $\epsilon \leq \Psi(X)/\Psi^* \leq \epsilon + \|\Psi\|_\infty/\Psi^*$*such that*

$$\begin{aligned}\Phi^{(j)}(U\backslash X) &\leq (1 - \epsilon/3r \cdot \frac{\Psi^*}{\Psi(U)}) \cdot \Phi^{(j)}(U),\\ \partial(U\backslash X) &\leq (1 - \epsilon/3r \cdot \frac{\Psi^*}{\Psi(U)}) \cdot \partial U + O_M(\pi^{1/p}(U)).\end{aligned}$$

We remark that one can obtain such sets X as in the corollaries above in time $O_M(t(|G[U]|))$.

A.2 Two bin-packing procedures

Proof of Lemma 15. We are given two colorings χ_0 and $\hat{\chi}_1$ of respective disjoint vertex sets W_0 and W_1 with $W_0 \cup W_1 = W$. Let $w^* := w(W)/k$ denote the average weight of a k-coloring of W. Similarly, let $w_0^* := w(W_0)/k$ and $w_1^* := w(W_1)/k$ be the average weight of k-colorings in W_0 and W_1, respectively. By our preconditions, we have $w\chi_0^{-1}(i) = w_0^* + O(\|w\|_\infty)$ and $w\hat{\chi}_1^{-1}(i) = w_1^* + O(\|w\|_\infty)$. So it holds for every color class $\chi_0^{-1}(i) \cup \chi_1^{-1}(i)$ of the direct sum $\chi_0 \oplus \hat{\chi}_1$,

$$w\chi_0^{-1}(i) + w\hat{\chi}_1^{-1}(i) = w_0^* + w_1^* + O(\|w\|_\infty) = w^* + O(\|w\|_\infty). \tag{19}$$

In the following, we write more conveniently $w_1(i) := w\hat{\chi}^{-1}(i)$ for the weight of a color class in coloring $\hat{\chi}_1$. We have the precondition $w_1(i) \leq w^* - \|w\|_\infty$.

We proceed in two phases to transform χ_0 into a coloring $\tilde{\chi}_0$ with the direct sum $\tilde{\chi}_0 \oplus \hat{\chi}_1$ being almost strictly balanced, i.e., $|w\tilde{\chi}_0^{-1}(i) + w_1(i) - w^*| \leq 2\|w\|_\infty$ for all $i \in [k]$. We start with $\tilde{\chi}_0 = \chi_0$. In color classes of $\tilde{\chi}_0$, we uncolor parts $X \subseteq W_0$ with $\|w\|_\infty \leq w(X) \leq 2\|w\|_\infty$, until the maximum of $w\tilde{\chi}_0^{-1}(i) + w_1(i)$ over all $i \in [k]$ is at most the average weight w^*. Then, we re-assign the previously uncolored parts X to color classes in a greedy manner. It follows that the direct sum $\tilde{\chi}_0 \oplus \hat{\chi}_1$ is almost strictly balanced.

Furthermore, since every considered part X has weight between $\|w\|_\infty$ and $2\|w\|_\infty$, we can infer from equation (19) that every color class of $\tilde{\chi}_0$ receives or emits only a constant number of parts. From this observation it shall follow that the maximum splitting cost of $\tilde{\chi}_0$ is at most proportional to the maximum splitting cost of χ_0. Similarly, the maximum boundary cost of $\tilde{\chi}_0$ is in $O(\|\partial\chi_0^{-1}\|_\infty + \|\pi\chi_0^{-1}\|_\infty^{1/p})$.

We now give the the full details of the proof. The procedure below transforms χ_0 into coloring $\tilde{\chi}_0$.

Procedure BinPack1 (coloring $\chi_0\colon W_0 \to [k]$, weight function $w_1\colon [k] \to \mathbb{R}_+$)

// *Precondition:* $w_1(i) \leq w^* - \|w\|_\infty$ *for all* $i \in [k]$

(1.) Start with $\tilde{\chi}_0 \leftarrow \chi_0$, and *Buffer* $\leftarrow \emptyset$.

(2.) As long as there exists a color class $U = \tilde{\chi}_0^{-1}(i)$ with $w(U) + w_1(i) > w^*$,
compute a splitting set $X \subseteq U$

with $\|w\|_\infty \le w(X) \le 2\|w\|_\infty$ and $\partial_U X \le \pi^{1/p}(U)$,
uncolor all vertices in X,
and update *Buffer* $\leftarrow$ *Buffer* $\cup \{X\}$.

(3.) As long as there is a color class $U = \tilde{\chi}_0^{-1}(i)$ with $w(U) + w_1(i) < w^* - 2\|w\|_\infty$
choose a part $X \in$ *Buffer*,
paint all vertices in X with color i,
and update *Buffer* $\leftarrow$ *Buffer* $\setminus \{X\}$

(4.) For all remaining $X \in$ *Buffer*,
choose a color $i \in [k]$ with $w\tilde{\chi}_0^{-1}(i) + w_1(i) \le w^*$,
and paint all vertices in X with color i.

(5.) Return coloring $\tilde{\chi}_0$

First, we need to show that we really can perform the above steps as described. In particular, each set U selected in step (2.) must have weight at least $\|w\|_\infty$ so that a subset X of U with $w(X) \ge \|w\|_\infty$ can be found. Also, *Buffer* needs to be non-empty whenever there exists a color class with $w\tilde{\chi}_0^{-1}(i) + w_1(i) < w^* - 2\|w\|_\infty$ in step (3.). We also would have to show that there exists a color $i \in [k]$ with $w\tilde{\chi}_0^{-1}(i) + w_1(i) \le w^*$ in step (4.). But this fact is obvious since w^* is the average of $w\tilde{\chi}_0^{-1}(j) + w_1(j)$ over all $j \in [k]$.

The invariants (I) and (II) below establish the soundness of steps (2.) and (3.) in procedure BinPack1.

Claim 1. The following are invariants of the algorithm above.

(I) Every color class $U = \tilde{\chi}_0^{-1}(i)$ with $w(U) + w_1(i) > w^*$ has weight $\ge w^*$

(II) In step (3.) we have $w\tilde{\chi}_0^{-1}(i) + w_1(i) \le w^*$ for all colors $i \in [k]$.

Proof: The precondition $w_1(i) \le w^* - \|w\|_\infty$ yields $w(U) \ge \|w\|_\infty$ for all vertex sets $U \subseteq W_0$ with $w(U) + w_1(i) \le w^*$. So invariant (I) holds. Clearly, invariant (II) is valid directly after the completion of step (2). An iteration in step (3) maintains the invariant since $w(X) \le 2\|w\|_\infty$ and therefore $w(U \cup X) + w_1(i) \le w^*$ for all $U \subseteq W_0$ and $i \in [k]$ with $w(U) + w_1(i) < w^* - 2\|w\|_\infty$. So invariant (II) holds, too. ∎

From the following two claims, which are implied by the precondition $w\chi_0^{-1}(i)+w_1(i) = w^*+O(\|w\|_\infty)$ and the $w(X) \ge \|w\|_\infty$ for all considered parts X, we infer that the maximum splitting cost and the maximum boundary cost of $\tilde{\chi}_0$ are as required by the lemma, i.e., $\|\pi\tilde{\chi}_0^{-1}\|_\infty = O(\|\pi\chi_0^{-1}\|_\infty$ and $\|\partial\tilde{\chi}_0^{-1}\|_\infty = O(\|\partial\chi_0^{-1}\|_\infty + \|\pi\chi_0^{-1}\|_\infty^{1/p})$.

Claim 2. The class of each color $i \in [k]$ is changed at most a constant number of times in steps (2)-(3) of procedure BinPack1.

Claim 3. For each considered part X, we have $\pi(X) \le \|\pi\chi_0^{-1}\|_\infty$ and $\partial(X) \le \|\partial\chi_0^{-1}\|_\infty + O(\|\pi\chi_0^{-1}\|_\infty^{1/p})$.

The claims above also imply that the procedure BinPack1 can be implemented to run in time at most proportional to $t(|G[W_0]|)$. The total time for computing splitting sets is $O(t(|G[W_0]|))$ by Claim 2. Using an appropriate data structure, e.g., a stack, it takes constant time to select a color that satisfies a certain condition (like $w\tilde{\chi}_0^{-1}(i)+w_1(i) > w^*$).

Now its easy to see that the direct sum $\tilde{\chi}_0 \oplus \hat{\chi}_1$ is almost strictly balanced. By invariant (II), we have $w\tilde{\chi}_0^{-1}(i) + w\hat{\chi}_1^{-1}(i) \le w*$ for all colors $i \in [k]$. When step (3) is completed, the minimum of $w\tilde{\chi}_0^{-1}(i) + w\hat{\chi}_1^{-1}(i)$ is at least $w^* - 2\|w\|_\infty$. Since $w(X) \le 2\|w\|_\infty$ for all considered parts X, the maximum of $w\tilde{\chi}_0^{-1}(i) + w\hat{\chi}_1^{-1}(i)$ cannot become larger than $w^* + 2\|w\|_\infty$. □

Proof of Proposition 12. We need the following simple claim.

Claim 4. For any vertex set $W \subseteq V$ with weight at least $\|w\|_\infty/2$, there exists $X \subseteq W$ with $\partial_W X \leq \pi^{1/p}(W) + \Delta_c$ and $\|w\|_\infty/2 \leq w(X) \leq \|w\|_\infty$.

Proof: If there exists a vertex $x \in W$ with weight at least $\|w\|_\infty/2$, then we can choose $X := \{x\}$. Otherwise, we have $\|w_{|W}\|_\infty \leq \|w\|_\infty/2$ and so we can compute a splitting set $X \subseteq W$ with $\partial_W X \leq \pi^{1/p}(W)$ and $\|w_{|W}\|_\infty \leq w(X) \leq 2\|w_{|W}\|_\infty$. Then part X has weight between $\|w\|_\infty/2$ and $\|w\|_\infty$. ■

The procedure below shall compute a strictly balanced coloring $\hat{\chi}$ from an almost strictly balanced coloring χ such that $\|\partial\hat{\chi}^{-1}\|_\infty = O(\|\partial\chi^{-1}\|_\infty + \|\pi\chi^{-1}\|_\infty^{1/p} + \Delta_c)$. Let $w^* := \|w\|_1/k$ be the average weight of a k-coloring in V. We assume $w^* \geq \|w\|_\infty/2$. The somehow degenerate case $w^* < \|w\|_\infty/2$ can be handled similarly.

Procedure BinPack2 (almost strictly balanced coloring $\chi\colon V \to [k]$)

// *Precondition:* $w^* \geq \|w\|_\infty/2$

(1.) Start with $\hat{\chi} \leftarrow \chi$, and *Buffer* $\leftarrow \emptyset$.

(2.) As long as there exists a color class $U = \hat{\chi}^{-1}(i)$ with $w(U) > w^*$,
 compute a splitting set $X \subseteq U$ as in Claim 1
 uncolor all vertices in X,
 and update *Buffer* $\leftarrow$ *Buffer* $\cup \{X\}$.

(3.) As long as there is a color class $U = \hat{\chi}^{-1}(i)$ with $w(U) < w^* - (1 - 1/k)\|w\|_\infty$
 choose a part $X \in$ *Buffer*,
 paint all vertices in X with color i,
 and update *Buffer* $\leftarrow$ *Buffer* $\setminus \{X\}$

(4.) For all remaining $X \in$ *Buffer*,
 choose a color $i \in [k]$ with $w\hat{\chi}^{-1}(i) \leq w^* - w(X)/k$,
 and paint all vertices in X with color i.

(5.) Return strictly balanced coloring $\hat{\chi}$

Similar to Lemma 15, in step (3) the invariant $w\hat{\chi}^{-1}(j) \leq w^* + \|w\|_\infty/k$ holds for all colors $j \in [k]$. Hence if there exists a color $i \in [k]$ in step (3) with $w\hat{\chi}^{-1}(i) < w^* - (1 - 1/k)\|w\|_\infty$, then the total weight of currently colored vertices is less than $(k-1)(w^* + \|w\|_\infty^*/k) + w^* - (1 - 1/k)\|w\|_\infty = \|w\|_1$. So there are vertices uncolored and *Buffer* must be non-empty.

In step (4) we can choose a color $i \in [k]$ with color class of weight at most $w^* - w(X)/k$ since the vertices in X are uncolored and so $w^* - w(X)/k$ is at least the average weight of the current coloring $\hat{\chi}$.

Since χ is almost strictly balanced and each considered part X has weight at least $\|w\|_\infty/2$, the class of a color changes at most constant number of times. So for each considered part X, we have $\partial X \leq \|\partial\chi^{-1}\|_\infty + O(\|\pi\chi^{-1}\|_\infty^{1/p} + \Delta_c)$ by Claim 1. And therefore the returned coloring $\hat{\chi}$ satisfies $\|\partial\hat{\chi}^{-1}\|_\infty \leq \|\partial\chi^{-1}\|_\infty + O(\|\pi\chi^{-1}\|_\infty^{1/p} + \Delta_c)$.

The procedure BinPack2 can be implemented to run in time $O(t(|G|) + k\log k)$. The colors in step (4) can be selected using a heap data structure, since we can choose in every iteration the color with class of minimum weight. This operation takes time $O(\log k)$, since

$O(k)$ parts are generated in step (2). The argument for the running time of the remaining steps is analogous to the proof of Lemma 15. □

A.3 Balanced Separators and Tight Examples

In this section we elaborate on how splitting sets are related to the more common notion of balanced separators. Specifically, we show that both notions are equivalent for bounded-degree graphs. Based on these results we show lower bounds for the min-max boundary decomposition cost that are essentially proportional to the upper bounds given by Theorem 4.

Definition 34 (Balanced Separation). A *separation* of a graph $G = (V, E)$ is a pair (A, B) of vertex sets with $A \cup B = V$ such that no edge of G joins $A \setminus B$ and $B \setminus A$.

A separation is *balanced* with respect to weights $w \colon V \to \mathbb{R}_+$ if the weight of both $A \setminus B$ and $B \setminus A$ is at most two third of the total weight, i.e., $\max\{w(A \setminus B), w(B \setminus A)\} \leq 2/3 \cdot \|w\|_1$. The *cost* of a separation (A, B) with respect to a cost function $\tau \colon V \to \mathbb{R}_+$ is given by $\tau(A \cap B)$. A vertex set $S \subseteq V$ is called *balanced separator* if $S = A \cap B$ for a balanced separation (A, B).

Similar to the splittability σ_p of a graph we can define its "separability".

Definition 35 (Separability, Separator Theorem). The *p-separability* of G with vertex costs τ is the minimum cost of a balanced separation in a subgraph of G relative to the p-norm of the subgraph's vertex costs, where the subgraph and its weights are worst possible, i.e.,

$$\beta_p(G, \tau) := \max_{W \subseteq V} \sup_{w \colon W \to \mathbb{R}_+} \min_{(A,B)} \tau(A \cap B) / \|\tau_{|W}\|_p$$

where the minimum is over all w-balanced separations (A, B) of $G[W]$.

A family $\mathcal{G}$ of pairs (G, τ) has a *p-separator theorem* if there exists a constant $C_{\mathcal{G}}$ such that the p-separability of G with vertex costs τ is at most $C_{\mathcal{G}}$ for all pairs (G, τ) in $\mathcal{G}$, i.e., if $\beta_{p|\mathcal{G}} = O_{\mathcal{G}}(1)$.

The remark below gives an overview of known and recent results about the separability and splittability of various graph classes.

Remark 36 (cf. [8]). For unit vertex costs,

- *planar graphs* [5] have $\beta_2 = O(1)$,
- *graphs with genus g* [3] have $\beta_2 = O(\sqrt{g})$,
- *graphs excluding a clique of size h as minor* [1] have $\beta_2 = O(h^{3/2})$,
- *well-shaped meshes* in a d-dimensional space [9] have $\beta_{d/(d-1)} = O_d(1)$,
- d-dimensional *k-nearest neighbor graphs* [6] have $\beta_{d/(d-1)} = O_d(k^{1/d})$.

For arbitrary costs,

- *planar graphs* [2] have $\beta_2 = O(1)$,

– *d-dimensional grid graphs* have $\sigma_{d/(d-1)} = O(d \cdot \log^{1/d} \phi)$, where ϕ is the fluctuation of the edge costs, i.e., the ratio of the maximum cost to the minimum (positive) cost (cf. Section 6).

The following lemma relates the notions of splittability and separability. Since we defined splittability in terms of edge costs and separability in terms of vertex costs, we need to translate between edge costs and vertex costs. Let $G = (V, E)$ be a graph with edge costs $c\colon E \to \mathbb{R}_+$. A natural choice of vertex costs $\tau\colon V \to \mathbb{R}_+$ corresponding to c is given by $\tau(v) := c(\delta(v))$ for each vertex $v \in V$. Then for every separation (A, B), the boundary cost $c(\delta(U))$ of any vertex set U with $A \backslash B \subseteq U \subseteq A$ is no more than $\tau(A \cap B)$. On the other hand, we want to be able to construct from vertex sets $U' \subseteq V$ separations (A', B') with $U' \subseteq A'$ and cost $\tau(A', B')$ proportional to $c(\delta(U'))$. For this, we require that the *local fluctuation* $\phi_\ell(c) := \max_{u \in e \in E} \tau(u)/c(e)$ is bounded. When we choose $B' := V \backslash U$ and A' to be the set of vertices reachable from U' by at most one edge, then the separation (A', B') has cost at most $\tau(A' \cap B') \leq 2\phi_\ell(c) \cdot c(\delta(U'))$, since every vertex $A' \cap B'$ is an endpoint of an edge in $\delta(U')$.

Notice that for the case of unit edge costs, the local fluctuation $\phi_\ell(\mathbb{1})$ equals the maximum degree Δ of G.

In the following, we shall see that $\sigma_p(G, c)$ is proportional to $\beta_p(G, \tau)$ when both the maximum degree Δ and the local fluctuation ϕ_ℓ are bounded. The lemma's proof is a slight generalization of the proof in [5] for the fact that cheap balanced separations imply inexpensive separations (A, B) with both $w(A \setminus B)$ and $w(B \setminus A)$ at most $\|w\|_1/2$.

Lemma 37. *Let $G = (V, E)$ be a graph with edge costs $c\colon E \to \mathbb{R}_+$. Then*

$$\beta_p(G, \tau)/\phi_\ell(c) \overset{1.)}{\ll_p} \sigma_p(G, c) \overset{2.)}{\ll_p} \phi_\ell \cdot \Delta^{1/q} \cdot \beta_p(G, \tau)$$

where τ are vertex costs with $\tau(v) := c(\delta(v))$, $f \ll_p g$ is short for $f = O_p(g)$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. 1.) $\beta_p = O(\phi_\ell \cdot \sigma_p)$: Let W be a subset of V with weights $w\colon W \to \mathbb{R}_+$. We need to show that there exists a w-balanced separation (A, B) of cost $\tau(A \cap B) = O(\phi_\ell \cdot \sigma_p \cdot \|\tau_{|W}\|_p)$.

If $w(v) > \|w\|_1/3$ for some vertex $v \in W$ then $(\{v\}, W)$ is a w-balanced separation of cost $\tau(v) \leq \|\tau_{|W}\|_p$.

So we can assume $\|w\|_\infty \leq \|w\|_1/3$. Then let $U \subseteq W$ be a splitting set with $\partial_W U \leq \sigma_p \cdot \|c_{|W}\|_p$ and $1/3 \cdot \|w\|_1 \leq w(U) \leq 1/3 \cdot \|w\|_1 + \|w\|_\infty$. Our assumption ensures $w(U) \leq 2/3 \cdot \|w\|_1$. Let $X \subseteq W$ contain the endpoints of the edges in the cut $C := \delta_{G[W]}(U)$. Now $(A, B) := (U \cup X, W \setminus U)$ is a balanced separation of $G[W]$. The cost of (A, B) satisfies

$$\begin{aligned} \tau(A \cap B) \leq \tau(X) &\leq \textstyle\sum_{\{u,v\} \in C} \tau(u) + \tau(v) = \textstyle\sum_{e \in C} 2\phi_\ell \cdot c_e \\ &= O(\phi_\ell \cdot c(C)) = O(\phi_\ell \cdot \sigma_p \cdot \|c_{|W}\|_p) \end{aligned}$$

And the p-norm of $c_{|W}$ is at most proportional to the p-norm of $\tau_{|W}$, since

$$2 \cdot \sum_{e \in E[W]} c_e^p = \sum_{v \in W} \sum_{e \in \delta(v)} c_e^p \leq \sum_{v \in W} (\tau(v))^p$$

So (A, B) is a balanced separation with cost at most proportional to $\phi_\ell \cdot \sigma_p \cdot \|\tau_{|W}\|_p$.

2.) $\sigma_p = O_p(\phi_\ell \cdot \Delta^{1/q} \cdot \beta_p)$: Let W be a subset of V with weights $w\colon W \to \mathbb{R}_+$ and splitting value w^*. We need to show that there exists w^*-splitting set $U \subseteq W$ of cost $\partial_W U = O_p(\phi_\ell \cdot \Delta^{1/q} \cdot \beta_p \cdot \|c_{|W}\|_p)$.

Similar to the p-splitting cost measure (cf. Definition 10), we define a weight function $\pi\colon V \to \mathbb{R}_+$ with $\pi(v) := (\beta_p \cdot c(\delta(v))^p = (\beta_p \cdot \tau(v))^p$. Then it holds $\pi(W') = \beta_p\|\tau_{|W'}\|_p$ for arbitrary vertex set $W' \subseteq V$. Thus, there exists balanced separators of cost $\pi^{1/p}(W') := (\pi(W'))^{1/p}$ in $G[W']$, and so we can call $\pi^{1/p}(W')$ the *separating cost* of $G[W']$.

The following procedure computes a separation (A_0, B_0) of $G[W]$ such that $w(A_0 \backslash B_0) \leq w^* - \|w\|_\infty/2 \leq w(A_0)$. The idea is to divide the vertices of $G[W]$ using a π-balanced separation (A, B) and then to proceed recursively on one of the graphs $G[A \setminus B]$ and $G[B \setminus A]$.

Procedure Split (vertex set $W \subseteq V$, $w^* \in \mathbb{R}_+$)

// *Precondition:* $0 \leq w^* \leq \|w_{|W}\|_1$

(1.) *Trivial case:* if $\pi(W) = 0$, then return separation (W, W)

(2.) Let (A, B) be π-balanced separation of $G[W]$ with cost $\tau(A \cap B) \leq \beta_p\|\tau_{|W}\|_p = \pi^{1/p}(W)$

(3.) If $w^* - \|w\|_\infty/2 < w(A \setminus B)$
then let $(A', B') = \text{Split}(A \setminus B, w^*)$ be separation of $G[A \setminus B]$
and return $(A_0, B_0) := (A' \cup (A \cap B), B' \cup B)$,

(4.) else if $w(A \setminus B) \leq w^* - \|w\|_\infty/2 \leq w(A)$
then return $(A_0, B_0) := (A, B)$,

(5.) else if $w(A) < w^* - \|w\|_\infty/2$
then let $(A', B') = \text{Split}(B \setminus A, w^* - w(A))$ be separation of $G[B \setminus A]$
and return $(A_0, B_0) := (A \cup A', B' \cup (A \cap B))$.

Since both $\pi(A \setminus B)$ and $\pi(B \setminus A)$ are at most $\frac{2}{3} \cdot \pi(W)$, it follows by induction on the size of the considered graph that $\tau(A_0 \cap B_0) = \tau(A \cap B) + \tau(A' \cap B') \leq \pi^{1/p}(W) \cdot \sum_{i=0}^{\infty}(\frac{2}{3})^{i/p} = O_p(\pi^{1/p}(W))$.

Without loss of generality we may assume that $G[W]$ is connected. (If $G[W]$ was not connected, we would need to apply Split only to one of the connected components of $G[W]$.) Hence it holds $\tau(v) = c(\delta(v)) \leq \phi_\ell \cdot c(\delta(v) \cap E(W)) = c(\delta_{G[W]}(v))$. Then we have by Hölder's inequality

$$\tau(v) = \phi_\ell \cdot \sum_{e \in \delta_{G[W]}(v)} c_e \leq \phi_\ell \cdot |\delta_{G[W]}(v)|^{1/q} \cdot (\sum_{e \in \delta_{G[W]}(v)} c_e^p)^{1/p} \leq \phi_\ell \cdot \Delta^{1/q} \cdot (\sum_{e \in \delta_{G[W]}(v)} c_e^p)^{1/p}.$$

So it holds $\pi(W) \leq \phi_\ell \cdot \Delta^{1/q} \cdot \sum_{v \in W} \sum_{e \in \delta_{G[W]}(v)} c_e^p$ and also $\pi^{1/p}(W) = O(\phi_\ell \cdot \Delta^{1/q} \cdot \beta_p \cdot \|c_{|W}\|_p)$. Thus we get $\tau(A_0 \cap B_0) = O_p(\phi_\ell \cdot \Delta^{1/q} \cdot \beta_p \cdot \|c_{|W}\|_p)$.

Given a separation (A_0, B_0) computed by Split$(W,\, w^*)$, we find a w^*-splitting set U of $G[W]$ as follows. Let $\{v_1, \ldots, v_h\} = A_0 \cap B_0$ be an enumeration of the separator $A_0 \cap B_0$, and let $i \in [h+1]$ be the largest index with $w(A \backslash B) + w(v_1) + \ldots + w(v_{i-1}) \leq w^* - \|w\|_\infty/2$. Then $U := A_0 \setminus B_0 \cup \{v_1, \ldots, v_{i-1}\}$ is w^*-splitting.

Since $A_0 \setminus B_0 \subseteq U \subseteq A_0$, the boundary cost of U cannot exceed $\tau(A_0 \cap B_0)$ and it holds $\partial U = O_p(\pi^{1/q}(W)) = O_p(\phi_\ell \cdot \Delta^{1/q} \cdot \beta_p \cdot \|c_{|W}\|_p)$ as required. □

We remark that the running time of procedure Split might be quite long. The reason is that the size of one of the graphs $G[A \setminus B]$ and $G[B \setminus A]$ could be almost as large as $|G[W]|$. However this issue can be resolved, by using for every second recursive call of the procedure (alternately), separations that are deg_W-balanced instead of π-balanced, where $deg_W : V \to \mathbb{R}_+$ assigns the degree in $G[W]$ to a vertex. Then both graphs $G[A \setminus B]$ and $G[B \setminus A]$ have size at most $\frac{2}{3}|G[W]|$. With this modification, Split(W, w^*) runs in time $O(t(|G[W]|))$, provided that one can find balanced separations of graphs $G[W']$ in time $t(|G[W']|)$ and $t\colon \mathbb{N} \to \mathbb{N}$ is a linear function.

Notice that the second part of the proof of Lemma 37 implies the following stronger statement. Let b be the maximum of $\min_{(A,B)} \tau(A \cap B)/\|\tau_{|W}\|_p$ over all sets $W \subseteq V$, where the minimum is over all π-balanced separations of $G[W]$. Then for arbitrary weights w, procedure Split can find w-balanced separations of G with cost at most $O_p(b \cdot \|\tau\|_p)$. Hence, we can draw the corollary below from the second part of the proof of Lemma 37 (cf. Split). This corollary observes that cheap balanced separators with respect to one "universal" measure imply cheap balanced separators with respect to arbitrary measures.

Corollary 38. *Let $G = (V, E)$ be a graph with vertex costs τ, and weights π be as in the proof of Lemma 37. Then,*

$$\beta_p(G, \tau) \ll_p \max_{W \subseteq V} \min_{(A,B)} \tau(A \cap B))/\|\tau_{|W}\|_p$$

where the minimum is over all π-balanced separations of $G[W]$.

Similar to the situation above, the first part of the proof of Lemma 37 implies the following stronger statement. Let $s := \max_{W \subseteq V} \partial^2_\infty(G[W], c_{|W})/\|c_{|W}\|_p \geq \sigma_p$. If we used strictly balanced 2-colorings instead of splitting sets, we could show $\beta_p = O(\phi_\ell \cdot s)$. Together with the second part, we obtain an upper bound on σ_p in terms of the min-max boundary decomposition cost ∂^k_2 for two colors:

Corollary 39. *For graphs $G = (V, E)$ with edge costs $c\colon E \to \mathbb{R}_+$, it holds*

$$\sigma_p(G, c) \ll_p \Delta^{1/q} \cdot \phi^2_\ell(c) \cdot \max_{W \subseteq V} \partial^2_\infty(G[W], c_{|W})/\|c_{|W}\|_p \leq \Delta^{1/q} \cdot \phi^2_\ell(c) \cdot \sigma_p(G, c).$$

In the remainder of this section we construct families of instances for which we can compute lower bounds on the min-max boundary decomposition cost. With these instances, we can argue that there is no way to improve the upper bound of Theorem 5, i.e., the bound is optimal with respect to the chosen parameters.

The idea is as follows. Let $G = (V, E)$ be a graph with edge costs c of bounded local fluctuation and weights w such that each balanced separation has cost $\Omega(b \cdot \|\tau\|_p)$. Then consider the graph $\tilde{G} = G^{(1)} \dot\cup \ldots \dot\cup G^{(\lfloor k/4 \rfloor)}$ consisting of $\lfloor k/4 \rfloor$ disjoint isomorphic copies $G^{(i)}$ of G. For a vertex $v \in V$, we write $v^{(i)}$ for the copy of v in $G^{(i)}$. Similarly, $e^{(i)}$ denotes the edge in $G^{(i)}$ that corresponds to an edge $e \in E$. We extend the costs and weights of G to the graph $\tilde{G}$ in the obvious way: $\tilde{c}(e^{(i)}) := c(e)$ and $\tilde{w}(v^{(i)}) := w(v)$. Now the claim is that every k-coloring χ of $\tilde{G}$ with $\|\tilde{w}\chi^{-1}\|_\infty \leq 2\|\tilde{w}\|_{avg}$ has average boundary cost $\Omega(b \cdot k^{-1/p} \cdot \|\tilde{c}\|_p)$.

Lemma 40. *Let $k \geq 4$ be an integer and $G = (V, E)$ be a graph with edge costs $c\colon E \to \mathbb{R}_+$ and vertex weights $w\colon V \to \mathbb{R}_+$. Suppose all w-balanced separations of G have cost at least $b \cdot \|\tau\|_p$ with respect to the vertex costs τ that correspond to c, i.e., $\tau(v) := c(\delta(v))$.*

Then for the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ that consists of $\lfloor k/4 \rfloor$ pairwise disjoint isomorphic copies of G, every k-coloring χ with roughly balanced weights, i.e., $\|\tilde{w}\chi^{-1}\|_\infty \leq 2\|\tilde{w}\|_{avg}$, has average boundary cost

$$\|\partial\chi^{-1}\|_{avg} = \Omega(b \cdot k^{-1/p} \cdot \|\tilde{c}\|_p / \phi_\ell(c)),$$

where $\tilde{c}\colon \tilde{E} \to \mathbb{R}_+$ and $\tilde{w}\colon \tilde{V} \to \mathbb{R}_+$ are the extensions of c and w to $\tilde{G}$, respectively.

Proof. We consider one of the isomorphic copies of G, say $G^{(i)} = (V^{(i)}, E^{(i)})$. Let $U_j := \chi^{-1}(j) \cap V^{(i)}$ be the set of vertices of $G^{(i)}$ with color j in coloring χ.

The coloring χ has maximum weight at most $2\|\tilde{w}\|_{avg}$, and the weight of $G^{(i)}$ is at least $4\|\tilde{w}\|_{avg}$. Hence for each U_j, it holds $w(U_j) \leq w(V^{(i)})/2$. Then we can (greedily) find a partition $\{R, B\}$ of the color set $[k]$ such that $\sum_{j \in R} w(U_j) \leq 2/3 \cdot w(V^{(i)})$ and $\sum_{j \in B} w(U_j) \leq 2/3 \cdot w(V^{(i)})$.

Let $U^* \subseteq V^{(i)}$ denote the set $\bigcup_{j \in R} U_j$ and $X \subseteq V^{(i)} \setminus U^*$ be the set vertices reachable from U^* by exactly one edge (of $\delta(U^*)$). So $(A, B) := (U^* \cup X, V^{(i)} \setminus U^*)$ is a w-balanced separation of $G^{(i)}$. By our precondition, we know $\tau(A \cap B) = \tau(X) \geq b \cdot \|\tau\|_p$. As in the first part of the proof of Lemma 37, we have $\partial U^* = \Omega(\tau(A \cap B)/\phi_\ell)$ and $\|\tau\|_p = \Omega(\|c\|_p)$.

Therefore, the total boundary cost of the coloring $\chi_{|V^{(i)}}$ satisfies $\|\partial\chi^{-1}_{|V^{(i)}}\|_1 \geq \partial(U^*) = \Omega(b \cdot \|c\|_p/\phi_\ell)$. Since i was arbitrary, the total boundary cost of χ

$$\|\partial\chi^{-1}\|_1 = \sum_{i=1}^{\lfloor k/4 \rfloor} \|\partial\chi^{-1}_{|V^{(i)}}\|_1 \geq \lfloor k/4 \rfloor \cdot \Omega(b \cdot \|c\|_p/\phi_\ell)$$

and hence the average boundary cost of χ is at least proportional to $b \cdot \|c\|_p/\phi_\ell$.

Since $\|\tilde{c}\|_p^p = (\sum_i \sum_{e \in E} c_e^p) = \lfloor \frac{k}{4} \rfloor \|c\|_p^p$, it holds $\|c\|_p = \Omega(\|\tilde{c}\|_p / k^{1/p})$ and thus we have $\|\partial\chi^{-1}\|_{avg} = \Omega(b \cdot k^{-1/p} \cdot \|\tilde{c}\|_p/\phi_\ell)$ as required. ☐

We get the following corollary from Lemma 40. Any graph for which we know a lower bound on the minimum cost balanced separation allows us to construct a "similar" graph with a lower bound on ∂^k_∞ that matches the upper bound from Theorem 5. We assume that the instance is well-behaved, i.e., the graph has bounded maximum degree Δ and the edge costs have bounded local fluctuation ϕ_ℓ.

Corollary 41. *Let $G = (V, E)$ be a well-behaved graph with edge costs $c\colon E \to \mathbb{R}_+$ and a p-separator theorem (with respect to vertex costs $\tau(v) := c(\delta(v))$ that correspond to the edge costs). Suppose there are weights $w\colon V \to \mathbb{R}_+$ such that $\|w\|_\infty \leq \|w\|_1/4$ and all w-balanced separations of G have cost $\Omega(\|\tau\|_p)$.*

Then for every positive multiple of 4, say k, there exists a well-behaved graph $\tilde{G}$ with edge costs $\tilde{c}$ and a p-separator theorem such that

$$\partial^k_\infty(\tilde{G}, \tilde{c}) = \Theta_p(\|\tilde{c}\|_p / k^{1/p} + \|\tilde{c}\|_\infty). \tag{20}$$

Also there are weights $\tilde{w}$ of $\tilde{G}$ such that every roughly $\tilde{w}$-balanced coloring has average boundary cost $\Omega(\|\tilde{c}\|_p / k^{1/p} + \|\tilde{c}\|_\infty)$.

Proof. Let $\tilde{G}$ and $\tilde{c}$ be as in Lemma 40. Observe that $\|\tilde{c}\|_\infty \leq \|c\|_p = O(\|\tilde{c}\|_p / k^{1/p})$.

The condition $\|w\|_\infty \leq \|w\|_1/4$ ensures $\|\tilde{w}\|_{avg} = \lfloor k/4 \rfloor \frac{\|w\|_1}{k} = \|w\|_1/4 \geq \|w\|_\infty$ and therefore every strictly $\tilde{w}$-balanced coloring of $\tilde{G}$ has maximum weight at most $\|\tilde{w}\|_{avg} + \|\tilde{w}\|_\infty \leq 2\|\tilde{w}\|_{avg}$.

So it follows from Lemma 40 that the average boundary cost (and also the maximum boundary cost) of every roughly or strictly balanced coloring is $\Omega(\|\tilde{c}\|_p/k^{1/p} + \|\tilde{c}\|_\infty)$.

The instance $(\tilde{G}, \tilde{c})$ is well-behaved and has a p-separator theorem, because it is a disjoint union of well-behaved instances with p-separator theorem. In fact, it is an easy consequence of procedure SPLIT (cf. Lemma 37) that $(\tilde{G}, \tilde{c})$ has a p-separator theorem.

Then it follows from Lemma 37 and the well-behavior of G that the p-splittability of $\tilde{G}$ is at most a constant. Since G has bounded maximum degree, it holds $\Delta_c(\tilde{G}) = O(\|\tilde{c}\|_\infty)$. So by Theorem 4 we have

$$\partial^k_\infty(\tilde{G}, \tilde{c}) = O_p(\|\tilde{c}\|_p/k^{1/p} + \|\tilde{c}\|_\infty). \qquad \square$$

We remark that it is a common assumption in previous work [4] to require from the considered graphs a p-separator theorem and bounded degree, i.e., that β_p and Δ are constants. We need the additional assumption that ϕ_ℓ is bounded, since we consider arbitrary costs instead of only unit-costs. Recall that $\phi_\ell = \Delta$ for unit costs.

The arguments in the proofs of Lemma 40 and Corollary 41 yield Theorem 5.